For the expert

For the expert in algebraic and differential topology who wants to get to the proof of the Morse Homology Theorem as quickly as possible we recommend the following:

4.1 (skip local formulas) → 6.2 – 6.3 (skip proof of Theorem 6.17) → 7.1 – 7.4 (skip 7.2)

Note that Smale's corollaries to the λ-Lemma (Theorem 6.17) found in Section 6.3 are **essential** for the proof of the Morse Homology Theorem given in Chapter 7. However, the results in Section 6.4 are **not needed** to prove the Morse Homology Theorem. There is an additional assumption on the Riemannian metric used to prove the results in Section 6.4 that is **not necessary** for the proof of the Morse Homology Theorem given in Chapter 7.

Notes on the proofs of the Morse Homology Theorem

There are at least six different ways to prove the Morse Homology Theorem:

1. Using the infinite dimensional techniques of Floer homology.
2. Using ideas from quantum mechanics and quantum field theory.
3. Using techniques from cobordism theory, i.e. handlebody decompositions.
4. Using Franks' Connecting Manifold Theorem (Theorem 6.41).
5. Using the Conley index and Conley's connection matrix (Chapter 7).
6. Using the manifold with corners structure of the compactified moduli spaces.

The first approach is the "modern" one found in the book **Morse Homology** by M. Schwarz [140]. In that book Schwarz constructs the Morse-Smale-Witten chain complex and shows that the resulting Morse homology theory satisfies the Eilenberg-Steenrod axioms. This proves that there exists (a non-explicit) isomorphism between Morse homology and singular homology. Using additional "modern" techniques involving pseudocycles and pseudohomology this isomorphism can be described explicitly [91] [141].

The second approach was introduced by Witten in his paper "Supersymmetry and Morse theory" [164], where he wrote down explicitly the formula for the Morse-Smale-Witten boundary operator (Definition 7.2). A mathematically rigorous exposition of Witten's approach appears in papers by Helffer and Sjöstrand [78].

The third approach is implicit in Section 6 of the book **Lectures on the h-cobordism Theorem** by J. Milnor [109]. However, the results in Milnor's book, as written, apply only to "self-indexing" Morse functions. The "self-indexing" assumption is not necessary for the Morse Homology Theorem.

The fourth approach uses the results of Franks [62], discussed in Section 6.4 of this book. This approach first appeared explicitly in [37]. Franks' results apply only in cases where the Riemannian metric is "compatible with the Morse charts" for the function (see Definition 6.30). In the literature, this additional assumption is sometimes expressed by saying that the Riemannian metric is "nice", the vector field is in "standard form" [62], or the gradient vector field is "Special Morse" [22]. Another similar assumption is that the function has "split Morse singularities" [21]. Although these sorts of extra assumptions greatly simplify some of the proofs in the theory, they are not needed for the Morse Homology Theorem.

The fifth approach, which we give in Chapter 7, is due to Floer [59] and Salamon [138]. There are several advantages to this approach. First of all, it does not require any superfluous assumptions on either the Morse function or the Riemannian metric, i.e. it works for any Morse-Smale function. Secondly, the proof produces an explicit isomorphism between the Morse homology groups and the singular homology groups (see the last line of the proof of Theorem 7.4). And finally, the proof uses only elementary "classical" techniques from algebraic topology and homotopy theory.

The sixth approach can be found in the papers by Burghelea and Haller [35], Laudenbach [22], Latour [100], Qin [126] [127], and the references therein. Using various techniques they prove that the moduli spaces of piecewise gradient flow lines of a Morse-Smale function are smooth manifolds with corners, the unstable manifolds give a CW-structure on the manifold, and the resulting CW-boundary operator can be identified directly with the Morse-Smale-Witten boundary operator.