Example 2.2 If $D^n = \{x \in \mathbb{R}^n | \ |x| \leq 1\}$ is the n-disk and ∂D^n is its boundary $S^{n-1} = \{x \in \mathbb{R}^n | \ |x| = 1\}$, then we have the following for $n > 0$.

$$H_k(S^n) = \begin{cases} \Lambda & \text{if } k = n \text{ or } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H_k(D^n, \partial D^n) = \begin{cases} \Lambda & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

When $k = 0$ this follows easily from the definition of singular homology. For $k > 0$ consider the long exact sequence of the pair (D^n, S^{n-1}).

$$\cdots \longrightarrow H_k(D^n) \xrightarrow{j_*} H_k(D^n, S^{n-1}) \xrightarrow{\delta_k} H_{k-1}(S^{n-1}) \xrightarrow{i_*} H_{k-1}(D^n) \longrightarrow$$

Since D^n is homotopic to a point, $H_k(D^n) = 0$ if $k > 0$, and hence we have $H_k(D^n, S^{n-1}) \approx H_{k-1}(S^{n-1})$ if $k > 1$. When $k = 1$ we have

$$0 \longrightarrow H_1(D^n, S^{n-1}) \xrightarrow{\delta_1} H_0(S^{n-1}) \xrightarrow{i_*} H_0(D^n) \longrightarrow 0,$$

and hence $H_1(D^n, S^{n-1}) \approx \ker(H_0(S^{n-1}) \xrightarrow{i_*} \Lambda)$ where i_* is surjective. Since $H_0(S^{n-1}) \approx \Lambda$ if $n > 1$ and $H_0(S^{n-1}) \approx \Lambda \oplus \Lambda$ if $n = 1$, we see that $H_1(D^n, S^{n-1}) = 0$ if $n > 1$ and $H_1(D^1, S^0) \approx \Lambda$. More explicitly, when $n = 1$ we have $H_0(S^0) = C_0(-1; \Lambda) \oplus C_0(1; \Lambda) \approx \Lambda \oplus \Lambda$ and $i_*([\sigma_{-1}], [\sigma_1]) = [\sigma_{-1}] + [\sigma_1] \in H_0(D^1)$. So, $\delta_1(H_1(D^1, S^0)) = \ker(H_0(S^0) \xrightarrow{i_*} \Lambda)$ is generated by $(-[\sigma_{-1}], [\sigma_1])$ where $\sigma_{-1} : \Delta^0 \to -1 \in D^1$ and $\sigma_1 : \Delta^0 \to 1 \in D^1$ are unique maps to points.

Now consider the long exact sequence of the pair (S^n, D^n_+) where D^n_+ is the upper hemisphere.

$$\cdots \longrightarrow H_k(D^n_+) \xrightarrow{i_*} H_k(S^n) \xrightarrow{j_*} H_k(S^n, D^n_+) \xrightarrow{\delta_k} H_{k-1}(D^n_+) \longrightarrow \cdots$$

Using the same reasoning as before we see that $H_k(S^n) \approx H_k(S^n, D^n_+)$ for $k > 1$. When $k = 1$ and $n > 0$ we have

$$\cdots \longrightarrow 0 \longrightarrow H_1(S^n) \xrightarrow{j_*} H_1(S^n, D^n_+) \xrightarrow{\delta_1} H_0(D^n_+) \xrightarrow{i_*} H_0(S^n) \longrightarrow 0$$

since $H_0(S^n, D^n_+) = 0$ (as seen from the definition of relative singular homology). Since $i_* : \Lambda \to \Lambda$ is an isomorphism, $\delta_1 = 0$, and hence j_* is also an isomorphism. Therefore, $H_k(S^n) \approx H_k(S^n, D^n_+)$ for all $k > 0$ if $n > 0$.

Finally, if U is a small open neighborhood of the north pole in S^n we can apply excision and the homotopy invariance of homology to conclude that

$$H_k(S^n, D^n_+) \approx H_k(S^n - U, D^n_+ - U) \approx H_k(D^n, S^{n-1})$$

for all k. Putting this all together we conclude that for $n > 0$ we have

$$H_k(S^n) \overset{k>0}{\approx} H_k(S^n, D^n_+) \approx H_k(D^n, S^{n-1}) \overset{k>1}{\approx} H_{k-1}(S^{n-1}).$$