S^{n-1}_τ is called a **relative attaching map**. As an immediate consequence of the CW-Homology Theorem we see that the homology of the chain complex $(\underline{C}_*(X, A; \Lambda), \underline{\partial}_*)$ is independent of the choice of the CW-structure on the space X.

Proof:

Since $\underline{\partial}_n : \underline{C}_n(X, A) \to \underline{C}_{n-1}(X, A)$ is defined as $\underline{\partial}_n = \Phi_{n-1} \circ \delta_* \circ \Psi_n$ we have $\underline{\partial}_{n-1} \circ \underline{\partial}_n = \Phi_{n-2} \circ \delta_* \circ \Psi_{n-1} \circ \Phi_{n-1} \circ \delta_* \circ \Psi_n = \Phi_{n-2} \circ (\delta_* \circ \delta_*) \circ \Psi_n = 0$ since $\delta_*^2 = 0$.

Let $\sigma \in \underline{C}_n(X, A)$ be a generator, i.e. an n-cell of X not in A. $\underline{\partial}_n(\sigma)$ is defined by taking $\sigma \mapsto f_{\sigma *}[D^n_\sigma] \in H_n(X^{(n)}, X^{(n-1)})$, applying the connecting homomorphism $\delta_n : H_n(X^{(n)}, X^{(n-1)}) \to H_{n-1}(X^{(n-1)})$, sending this into $H_{n-1}(X^{(n-1)}, X^{(n-2)})$ by j_*, and then applying Φ_{n-1}. Since δ_n is a natural homomorphism, the following diagram commutes

$$\begin{array}{ccc} H_n(D^n_\sigma, \partial D^n_\sigma) & \xrightarrow{\delta_n} & H_{n-1}(\partial D^n_\sigma) \\ \downarrow{\scriptstyle f_{\sigma *}} & & \downarrow{\scriptstyle f_{\partial\sigma *}} \\ H_n(X^{(n)}, X^{(n-1)}) & \xrightarrow{\delta_n} & H_{n-1}(X^{(n-1)}) \end{array}$$

and we see that $\delta_n(f_{\sigma *}[D^n_\sigma]) = f_{\partial\sigma *}[\partial D^n_\sigma]$. (See Example 2.2 and Lemma 2.11 When $n = 1$, $[\partial D^1_\sigma] = -[\sigma_{-1}] + [\sigma_1]$). Hence,

$$\begin{aligned} \underline{\partial}_n(\sigma) = \Phi_{n-1}(f_{\partial\sigma *}[\partial D^n_\sigma]) &= \sum_\tau \phi_{n-1}(p_{\tau *} f_{\partial\sigma *}[\partial D^n_\sigma])\tau \\ &= \sum_\tau \phi_{n-1}((p_\tau \circ f_{\partial\sigma})_*[\partial D^n_\sigma])\tau \\ &= \sum_\tau \phi_{n-1}(\deg(p_\tau \circ f_{\partial\sigma}) \cdot [S^{n-1}])\tau \\ &= \sum_\tau \deg(p_\tau \circ f_{\partial\sigma})\tau \end{aligned}$$

by the definition of the degree.

Now consider the exact sequence of the triple $X^{(n-2)} \subseteq X^{(n-1)} \subseteq X^{(n)}$.

$$\begin{array}{ccccccc} H_n(X^{(n-1)}, X^{(n-2)}) & \longrightarrow & H_n(X^{(n)}, X^{(n-2)}) & \longrightarrow & H_n(X^{(n)}, X^{(n-1)}) & \xrightarrow{\delta_*} & H_{n-1}(X^{(n-1)}, X^{(n-2)}) \\ \approx\downarrow{\scriptstyle \text{Lemma 2.11}} & & \approx\downarrow{\scriptstyle \text{Lemma 2.12}} & & \approx\downarrow{\scriptstyle \text{Lemma 2.11}} & & \approx\downarrow{\scriptstyle \text{Lemma 2.11}} \\ 0 & \longrightarrow & H_n(X^{(n)}, A) & \longrightarrow & \underline{C}_n(X, A) & \xrightarrow{\underline{\partial}_n} & \underline{C}_{n-1}(X, A) \end{array}$$

From this we deduce that $H_n(X^{(n)}, X^{(n-2)})$ can be identified with ker $\underline{\partial}_n$, that is

$$H_n(X^{(n)}, X^{(n-2)}) \approx H_n(X^{(n)}, A) \approx \ker \underline{\partial}_n = \underline{Z}_n(X, A).$$