To prove this we will need the following.

Lemma 3.9 *Let $f : U \to \mathbb{R}$ be a smooth function on an open set $U \subseteq \mathbb{R}^m$. For almost all $a = (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$, the function*

$$f_a(x) = f(x) - \sum_{j=1}^{m} x_j a_j$$

is a Morse function.

Proof:

Consider the smooth function $g : \mathbb{R}^m \to \mathbb{R}^m$ given by

$$g(x) = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_m}(x) \right).$$

Let $a \in \mathbb{R}^m$ be a regular value of g. By Sard's Theorem (see for instance Theorem II.6.2 of [32]), the set of regular values $a \in \mathbb{R}^m$ is dense in $\mathbb{R}^m$. Consider the function

$$f_a(x) = f(x) - \sum_{j=1}^{m} x_j a_j.$$

If $p \in \mathbb{R}^m$ is a critical point of f_a, then

$$(df_a)_p = g(p) - a = 0.$$

Since a is a regular value of g, dg_p is surjective, and hence it is invertible. The Hessian $H_p(f_a)$ is precisely dg_p, and thus p is non-degenerate.

□

Remark 3.10 We may choose the regular value a as close to $0 \in \mathbb{R}^n$ as we wish. Hence, the Morse function f_a may be made as close to f as we wish. For a global version of the preceding lemma see Theorem 5.27.

Proof of Theorem 3.8:

Let $(x_1, \ldots, x_r)$ be the coordinates of a point $x \in M \subseteq \mathbb{R}^r$. There is a neighborhood U of x on which some of the x_j's, say $(x_{j_1}, \ldots, x_{j_m})$, form local coordinates on M. Indeed, since $T_xM \hookrightarrow T_x\mathbb{R}^r$ is injective, its dual $T_x^*\mathbb{R}^r \to T_x^*M$ is surjective. Therefore, for some neighborhood U of x, $T^*M|_U$ is spanned by some linearly independent set $dx_{j_1}, \ldots, dx_{j_m}$. Consequently, $(x_{j_1}, \ldots, x_{j_m})$ are linearly independent and hence form a coordinate system on U.