Then we have ${}^tQSQ = A$ where $P(S) = Q \in GL_m(\mathbb{R})$ depends smoothly on S and $P(A) = I_{m\times m}$.

□

3.2 The gradient flow of a Morse function

Recall that an **inner product** on a vector space V over the field $\mathbb{R}$ is a bilinear function $< \cdot, \cdot >: V \times V \to \mathbb{R}$ which is **symmetric**, that is $< v, w > = < w, v >$ for all $v, w \in V$, and **non-degenerate**, i.e. if $v \neq 0$, then there exists some $w \neq 0$ such that $< v, w > \neq 0$. An inner product is said to be **positive definite** if and only if $< v, v >$ is strictly greater than zero for all $v \in V$. The tangent bundle T_*M over a smooth manifold M is a smooth vector bundle whose fiber at each point $x \in M$ is the tangent space T_xM, and a Riemannian metric g on T_*M is a smooth function that assigns to each $x \in M$ a positive definite inner product $< \cdot, \cdot >_x$ on T_xM. For more details concerning inner products and Riemannian metrics see for instance Chapter 9 of [149].

A non-degenerate inner product on a vector space V induces an isomorphism between V and its dual V^*, and hence, a Riemannian metric g on a smooth manifold M defines an isomorphism $\tilde{g} : T_*M \to T^*M$ between the tangent and cotangent bundles. For any vector field W, $\tilde{g}(W)$ is the unique 1-form such that for any vector field V

$$\tilde{g}(W)(V) = g(W, V).$$

Definition 3.17 *If $f : M \to \mathbb{R}$ is a smooth function on a Riemannian manifold (M, g), then the* ***gradient vector field*** *of f with respect to the metric g is the unique smooth vector field ∇f such that*

$$g(\nabla f, V) = df(V) = V \cdot f$$

for all smooth vector fields V on M, i.e. $\nabla f = \tilde{g}^{-1}(df)$. In particular,

$$(\nabla f) \cdot f = g(\nabla f, \nabla f) = ||\nabla f||^2.$$

Let $\varphi_t : M \to M$ be the local 1-parameter group of diffeomorphisms generated by $-\nabla f$ (the negative gradient), i.e.

$$\begin{aligned} \frac{d}{dt}\varphi_t(x) &= -(\nabla f)(\varphi_t(x)) \\ \varphi_0(x) &= x. \end{aligned}$$

(See for instance [82] Section 6.2 or [96] Section I.6.) The integral curve $\gamma_x : (a, b) \to M$ given by

$$\gamma_x(t) = \varphi_t(x)$$