is called a gradient **flow line**.

We have the following easy but important facts.

Proposition 3.18 *Every smooth function $f : M \to \mathbb{R}$ on a finite dimensional smooth Riemannian manifold (M, g) decreases along its gradient flow lines.*

Proof:

$$\begin{aligned} \frac{d}{dt} f(\gamma_x(t)) &= \frac{d}{dt}(f \circ \varphi_t(x)) \\ &= df_{\varphi_t(x)} \circ \frac{d}{dt}\varphi_t(x) \\ &= df_{\varphi_t(x)}(-(\nabla f)(\varphi_t(x))) \\ &= -\|(\nabla f)(\varphi_t(x))\|^2 \leq 0 \end{aligned}$$

□

Proposition 3.19 *Let $f : M \to \mathbb{R}$ be a Morse function on a finite dimensional compact smooth Riemannian manifold (M, g). Then every gradient flow line of f begins and ends at a critical point, i.e. for any $x \in M$, $\lim_{t\to+\infty} \gamma_x(t)$ and $\lim_{t\to-\infty} \gamma_x(t)$ exist, and they are both critical points of f.*

Proof:

Let $x \in M$ and let $\gamma_x(t)$ be the gradient flow line through x. Since M is compact, $\gamma_x(t)$ is defined for all $t \in \mathbb{R}$ (see for instance Section 6.2 of [82] or Corollary I.6.2 of [96]), and the image of $f \circ \gamma_x : \mathbb{R} \to \mathbb{R}$ is a bounded subset of $\mathbb{R}$. Hence by Proposition 3.18 we must have

$$\lim_{t\to\pm\infty} \frac{d}{dt} f(\gamma_x(t)) = \lim_{t\to\pm\infty} -\|(\nabla f)(\varphi_t(x))\|^2 = 0.$$

Let $t_n \in \mathbb{R}$ be a sequence with $\lim_{n\to\infty} t_n = -\infty$. The set $\{\gamma_x(t_n)\} \subseteq M$ is an infinite set of points in a compact manifold, and so it has an accumulation point q. The point q is a critical point of f since $\|(\nabla f)(\gamma_x(t_n))\| \to 0$ as $n \to \infty$, and by Lemma 3.2 we can pick a closed neighborhood U of q where q is the only critical point in U. If $\lim_{t\to-\infty} \gamma_x(t) \neq q$, then there is an open neighborhood $V \subset U$ of q and a sequence $\tilde{t}_n \in \mathbb{R}$ with $\lim_{n\to\infty} \tilde{t}_n = -\infty$ and $\gamma_x(\tilde{t}_n) \in U - V$. Thus, the sequence $\{\gamma_x(\tilde{t}_n)\}$ has an accumulation point in the compact set $U - V$ which, as above, must be a critical point of f. This contradicts the choice of U, and therefore, $\lim_{t\to-\infty} \gamma_x(t) = q$. A similar argument shows that $\lim_{t\to+\infty} \gamma_x(t) = p \in M$ for some critical point p.

□