and

$$c(t) - c(0) = \int_0^t \frac{d}{ds} c(s)\, ds = t,$$

i.e.

$$f(\Psi_t(x)) = c(t) = c(0) + t = f(\Psi_0(x)) + t = f(x) + t.$$

Suppose now that $f(x) = a$, then $c(t) = a + t$, and hence,

$$c(b - a) = a + (b - a) = b.$$

So, if $x \in f^{-1}(a)$ and $y = \Psi_{b-a}(x)$, then $f(y) = b$.

Consider the diffeomorphism $\Psi_{b-a} : M \to M$. We just saw that any $x \in M$ with $f(x) = a$ is taken to y with $f(y) = b$, and clearly if $f(x) < a$, then for $y = \Psi_{b-a}(x)$ we have $f(y) < b$. Therefore, Ψ_{b-a} maps M^a into M^b and its inverse $\Psi_{a-b} = \Psi_{b-a}^{-1}$ maps M^b into M^a. Hence, M^a and M^b are diffeomorphic.

Now consider the 1-parameter family of maps $r_t : M^b \to M^b$ given by

$$r_t(x) = \begin{cases} x & \text{if } f(x) \le a \\ \Psi_{t(a-f(x))}(x) & \text{if } a \le f(x) \le b. \end{cases}$$

This family is continuous since for $a = f(x)$ we have $\Psi_{t(a-f(x))}(x) = \Psi_0(x)$ $= x$. Since r_0 is the identity map and r_1 is a retraction from M^b to M^a, we see that M^a is a deformation retract of M^b.

For the second part of the theorem, define $F : f^{-1}(a) \times [a, b] \to f^{-1}([a, b])$ by $F(x, t) = \Psi_{t-a}(x)$. The above computation shows that for any $x \in f^{-1}([a, b])$ we have $\Psi_{a-f(x)}(x) \in f^{-1}(a)$. Thus, $F(\Psi_{a-f(x)}(x), f(x)) = x$, and F is surjective. Since f is increasing along the flow lines of X, the function $f(F(x, t))$ is increasing in t. Hence F is injective, and F is an immersion because gradient lines are transverse to level sets. Therefore, F is a diffeomorphism.

□

Corollary 3.21 *Let M be a compact smooth manifold with boundary $\partial M = A \amalg B$, i.e. $A \cap B = \emptyset$. Suppose there exists a C^∞ function $f : M \to [0, 1]$ with no critical points such that $f(A) = 0$ and $f(B) = 1$. Then M is diffeomorphic to $A \times [0, 1] \approx B \times [0, 1]$.*

Proof:

This follows from the second part of the theorem since $f^{-1}([0, 1]) = M$.

□