Corollary 3.22 (Reeb) *If M is a compact smooth manifold without boundary of dimension m admitting a Morse function $f : M \to \mathbb{R}$ with only 2 critical points, then M is homeomorphic to the m-sphere S^m.*

Proof:

Let $f : M \to \mathbb{R}$ be a smooth function with exactly 2 critical points. Since M is compact, f attains a maximum at some point $p_+ \in M$ and a minimum at some point $p_- \in M$. Let $f(p_+) = z_+$ and $f(p_-) = z_-$. The Morse Lemma (Lemma 3.11) implies that there exists an open neighborhood U_+ of p_+ and coordinates $(u_1, \ldots, u_m)$ on which $f(x) = z_+ - (u_1^2 + \cdots + u_m^2)$. Hence for some $b < z_+$ but close to z_+, the set $D_+ = f^{-1}([b, z_+]) = \{(u_1, \ldots, u_m) | u_1^2 + \cdots + u_m^2 \leq z_+ - b\}$ is diffeomorphic to the closed m-disk D^m. Similarly, for some $a > z_-$ but close to z_- the set $D_- = f^{-1}([z_-, a])$ is also diffeomorphic to D^m. By Corollary 3.21, $f^{-1}([a, b])$ is diffeomorphic to $S^{m-1} \times [0, 1]$. To get the homeomorphism with S^m we put together D_-, $S^{m-1} \times [0, 1]$, and D_+.

Let $q_\pm$ be the north and south pole of S^m respectively, and let $B_\pm$ be disjoint neighborhoods of $q_\pm \in S^m$ diffeomorphic to D^m so that $C = S^m - \text{Int}(B_+ \cup B_-) \approx S^{m-1} \times [0, 1]$ and $\partial C = \partial B_+ \cup \partial B_-$. Let $h_+ : D_+ \to B_+ \approx D^m$ be the diffeomorphism given by the Morse Lemma. Extend $h_+|_{\partial D_+} : \partial D_+ \to \partial B_+$ to a diffeomorphism from $\partial D_+ \times [0, 1] \to \partial B_+ \times [0, 1]$. Since $\partial D_+ \times [0, 1]$ is diffeomorphic to $f^{-1}([a, b])$, this gives an extension of h_+ to a homeomorphism

$$h : D_+ \cup f^{-1}([a, b]) \to B_+ \cup C.$$

Let g_0 be the restriction of h to ∂D_-,

$$g_0 : \partial D_- \approx S^{m-1} \to \partial B_- \approx S^{m-1}$$

and extend g_0 radially to a homeomorphism $g : D_- \approx D^m \to B_- \approx D^m$, i.e.

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ \|x\| g_0(\frac{x}{\|x\|}) & \text{if } x \neq 0. \end{cases}$$

Putting together h and g we get a homeomorphism from M to S^m.

□

Remark 3.23 The homeomorphism in Corollary 3.22 may fail to be a diffeomorphism. We refer the reader to Chapter X of [96] for more details. In particular, see Section X.7 of [96] for a good historical introduction to the classification of smooth structures on S^m.