and define $h_1 : X \cup_{f_1} D^k \to X \cup_{f_0} D^k$ by $h_1(x) = x$ if $x \in X$ and for all $u \in S^{k-1}$

$$h_1(f_1(ru)) = \begin{cases} f_0(2ru) & \text{if } 0 \leq 2r \leq 1 \\ f_{2r-1}(u) & \text{if } 1 \leq 2r \leq 2. \end{cases}$$

It is easy to verify that h_0 and h_1 are single valued and hence continuous. We have for all $u \in S^{k-1}$

$$(h_1 \circ h_0)(f_0(ru)) = \begin{cases} h_1(f_1(2ru)) & \text{if } 0 \leq 2r \leq 1 \\ h_1(f_{2-2r}(u)) & \text{if } 1 \leq 2r \leq 2. \end{cases}$$

Since $h_1(x) = x$ for all $x \in X$ it follows that for all $u \in S^{k-1}$

$$(h_1 \circ h_0)(f_0(ru)) = \begin{cases} f_0(4ru) & \text{if } 0 \leq 4r \leq 1 \\ f_{4r-1}(u) & \text{if } 1 \leq 4r \leq 2 \\ f_{2-2r}(u) & \text{if } 1 \leq 2r \leq 2. \end{cases}$$

Let $\xi_t : X \cup_{f_0} D^k \to X \cup_{f_1} D^k$ be the homotopy which is defined by $\xi_t(x) = x$ for all $x \in X$ and for all $u \in S^{k-1}$

$$\xi_t(f_0(ru)) = \begin{cases} f_0((4-3t)ru) & \text{if } 0 \leq r \leq \frac{1}{4-3t} \\ f_{(4-3t)r-1}(u) & \text{if } \frac{1}{4-3t} \leq r \leq \frac{2-t}{4-3t} \\ f_{\frac{1}{2}(4-3t)(1-r)}(u) & \text{if } \frac{2-t}{4-3t} \leq r \leq 1. \end{cases}$$

It is easy to verify that ξ_t is single valued and hence continuous, $\xi_0 = h_1 \circ h_0$, and $\xi_1 = 1$. A homotopy $\eta_t : X \cup_{f_1} D^k \to X \cup_{f_0} D^k$ such that $\eta_0 = h_0 \circ h_1$ and $\eta_1 = 1$ is defined by replacing f_0 with f_1 and f_λ with $f_{1-\lambda}$ in the above expression for ξ_t where $\lambda = (4-3t)r - 1$ or $(4-3t)(1-r)/2$.

□

Lemma 3.30 (P. Hilton [107]**)** *Let X be a topological space and let*

$$f : S^{k-1} \to X$$

be an attaching map. Any homotopy equivalence $h : X \to Y$ extends to a homotopy equivalence

$$H : X \cup_f D^k \to Y \cup_{h \circ f} D^k.$$

Proof:

Define $H : X \cup_f D^k \to Y \cup_{h \circ f} D^k$ by

$$H(x) = \begin{cases} h(x) & \text{if } x \in X \\ x & \text{if } x \in D^k. \end{cases}$$