where each homotopy equivalence extends the previous one. Let X denote the union $\bigcup_i X_i$ with the direct limit topology, and let $h : M \to X$ be the direct limit map. Then h induces isomorphisms of homotopy groups in all dimensions, and hence, h is a homotopy equivalence (see for instance Theorem 1 of [162]).

□

Remark 3.31 Theorem 1 of [162] can be stated as follows. Let X and Y be connected topological spaces that are dominated by CW-complexes. Then a map $h : X \to Y$ is a homotopy equivalence of and only if $f_n : \pi_n(X) \to \pi_n(Y)$ is an isomorphism for every n such that $1 \leq n \leq N$ where $N = \max(\Delta X, \Delta Y) \leq \infty$.

A space P **dominates** a space X if and only if there are maps $\sigma : X \to P$ and $\sigma' : P \to X$ such that $\sigma' \circ \sigma \simeq 1$. A CW-complex X is dominated by itself, and a finite dimensional manifold M is dominated by the CW-complex consisting of a tubular neighborhood of M embedded in some Euclidean space. If X is dominated by a CW-complex of finite dimension, then ΔX denotes the minimum dimension of all CW-complexes that dominate X. If none of the CW-complexes that dominate X have finite dimension, then $\Delta X = \infty$.

Remark 3.32 There is a version of Theorem 3.28 called the **Handle Presentation Theorem** that keeps track of the differentiable structure. See Chapter VII of [96] for more details.

3.4 The Morse Inequalities

Let M be a compact smooth manifold and F a field. Then $H_k(M; F)$ is a finite dimensional vector space over F, and the k^{th} **Betti number** of M, denoted $b_k(F)$, is defined to be the dimension of $H_k(M; F)$. Similarly, if $F = \mathbb{Z}$ then $H_k(M; \mathbb{Z})$ modulo its torsion subgroup is a finitely generated free $\mathbb{Z}$-module, and $b_k(\mathbb{Z})$ is defined to be the rank of this finitely generated free $\mathbb{Z}$-module. We will write b_k for $b_k(F)$ when $F = \mathbb{Z}$ or F is understood from the context.

Let $f : M \to \mathbb{R}$ be a Morse function on M, and let ν_k be the number of critical points of f of index k for all $k = 0, \ldots, m$. As a consequence of Theorem 2.15 and Theorem 3.28 we have

$$\nu_k \geq b_k(F)$$

for all $k = 0, \ldots, m$ since $\nu_k = \text{rank } \underline{C}_k(X; F)$ and $H_k(M; F)$ is a quotient of this module. These inequalities are known as the **weak Morse inequalities**.