the Morse inequalities (that does not rely on Theorem 3.28) see Section 5 of [107].

In the following we will first prove Lemma 3.43 which states that the Morse inequalities are equivalent to the polynomial Morse inequalities. The reader should note that the proof of Lemma 3.43 shows explicitly the connection between the inequalities in part (a) of Theorem 3.33 and the coefficients of the polynomial $R(t)$ in Theorem 3.36. Next we will prove both Theorem 3.33 and Theorem 3.36. Of course, Lemma 3.43 shows that it would be sufficient to prove either one of these theorems. However, both proofs are of interest, and so we complete this circle of ideas by including separate proofs for both Theorem 3.33 and Theorem 3.36.

Lemma 3.43 *Theorem 3.33 is equivalent to Theorem 3.36.*

Proof:

By part (b) of Theorem 3.33 we have

$$M_{-1}(f) = \sum_{k=0}^{m}(-1)^k \nu_k = \sum_{k=0}^{m}(-1)^k b_k(F) = P_{-1}(M).$$

Thus $M_t(f) - P_t(M)$ is divisible by $1+t$, and $M_t(f) = P_t(M) + (1+t)R(t)$ for some polynomial $R(t) = \sum_{n=0}^{m-1} r_n t^n$. It's clear that $r_n \in \mathbb{Z}$ for all $n = 0, \ldots, m-1$ since both $M_t(f)$ and $P_t(M)$ have integer coefficients, and it remains to show that $r_n \geq 0$ for all $n = 0, \ldots, m-1$. We will show that this is equivalent to the inequalities in part (a) of Theorem 3.33.

To see this, first note that $M_t(f) = P_t(M) + (1+t)R(t)$ implies that

$$\nu_0 = b_0(F) + r_0.$$

Next, note that $\nu_1 = b_1(F) + r_1 + r_0$, and so $\nu_1 = b_1(F) + r_1 + \nu_0 - b_0(F)$, i.e.

$$\nu_1 - \nu_0 = b_1(F) - b_0(F) + r_1.$$

Continuing in this fashion we see that

$$\nu_n - \nu_{n-1} + \cdots + (-1)^n \nu_0 = b_n(F) - b_{n-1}(F) + \cdots + (-1)^n b_0(F) + r_n$$

for all $n = 0, \ldots, m-1$. Thus, the inequalities in part (a) of Theorem 3.33 imply that $r_n \geq 0$ for all $n = 0, \ldots, m-1$, and Theorem 3.33 implies Theorem 3.36.

Now assume that $M_t(f) = P_t(M) + (1+t)R(t)$ where $R(t) = \sum_{n=0}^{m-1} r_n t^n$ is a polynomial with non-negative integer coefficients. As above, this implies that

$$\nu_n - \nu_{n-1} + \cdots + (-1)^n \nu_0 = b_n(F) - b_{n-1}(F) + \cdots + (-1)^n b_0(F) + r_n$$