for all $n = 0, \ldots, m-1$. Since $r_n \geq 0$ for all $n = 0, \ldots, m-1$ we recover the inequalities in part (a) of Theorem 3.33. To recover part (b) we simply put $t = -1$ in the equation $M_t(f) = P_t(M) + (1+t)R(t)$. Therefore, Theorem 3.36 implies Theorem 3.33.

□

We will now show that Theorem 3.33 is an easy consequence of Theorem 3.28 and the Euler-Poincaré Theorem. Recall that a finitely generated chain complex $(\underline{C}_*, \underline{\partial}_*)$ is a chain complex such that $\underline{C}_k$ is a finitely generated abelian group for all $k \in \mathbb{Z}_+$, and $\underline{C}_k = 0$ except for a finite set of integers.

Theorem 3.44 (Euler-Poincaré Theorem) *Let $(\underline{C}_*, \underline{\partial}_*)$ be a finitely generated chain complex, and assume that $\underline{C}_k = 0$ for all $k > m$. Let $c_k = \text{rank } \underline{C}_k$ and $b_k = \text{rank } H_k(\underline{C}_*, \underline{\partial}_*)$ for all $k = 0, \ldots, m$. Then,*

$$\sum_{k=0}^{m} (-1)^k c_k = \sum_{k=0}^{m} (-1)^k b_k.$$

Proof:
The exact sequence

$$0 \rightarrow \text{ker } \underline{\partial}_k \rightarrow \underline{C}_k \stackrel{\underline{\partial}_k}{\rightarrow} \text{im } \underline{\partial}_k \rightarrow 0$$

shows that $c_k = \text{rank ker } \underline{\partial}_k + \text{rank im } \underline{\partial}_k$ for all $k = 0, \ldots, m$. Similarly,

$$0 \rightarrow \text{im } \underline{\partial}_{k+1} \rightarrow \text{ker } \underline{\partial}_k \rightarrow H_k(\underline{C}_*, \underline{\partial}_*) \rightarrow 0$$

shows that $\text{rank ker } \underline{\partial}_k = \text{rank im } \underline{\partial}_{k+1} + b_k$, and hence

$$\text{rank ker } \underline{\partial}_k = c_k - \text{rank im } \underline{\partial}_k = \text{rank im } \underline{\partial}_{k+1} + b_k$$

for all $k = 0, \ldots, m$. Thus,

$$\sum_{k=0}^{m} (-1)^k (c_k - \text{rank im } \underline{\partial}_k) = \sum_{k=0}^{m} (-1)^k (\text{rank im } \underline{\partial}_{k+1} + b_k)$$

which implies that

$$\sum_{k=0}^{m} (-1)^k c_k = \sum_{k=0}^{m} (-1)^k b_k.$$

□

Now, let $f : M \rightarrow \mathbb{R}$ be a Morse function on a compact smooth manifold M, and let X be the associated CW-complex given by Theorem 3.28. Let