The Morse-Bott Lemma

The following is the analogue of Lemma 3.11 for Morse-Bott functions. As a consequence of this lemma we see that there is a well-defined "Morse-Bott index" $\lambda_C \in \mathbb{Z}_+$ associated to a connected critical submanifold C.

Lemma 3.51 (Morse-Bott Lemma) *Let $f : M \to \mathbb{R}$ be a Morse-Bott function and $C \subseteq \mathrm{Cr}(f)$ a connected component. For any $p \in C$ there is a local chart of M around p and a local splitting $\nu_* C = \nu_*^- C \oplus \nu_*^+ C$, identifying a point $x \in M$ in its domain to (u, v, w) where $u \in C$, $v \in \nu_*^- C$, $w \in \nu_*^+ C$ such that within this chart f assumes the form*

$$f(x) = f(u, v, w) = f(C) - |v|^2 + |w|^2.$$

Proof:

Let $C \subseteq M$ be an n-dimensional connected critical submanifold of f. By replacing f with $f - c$, where c is the common value of f on the critical submanifold C, we may assume that $f(p) = 0$ for all $p \in C$. Let $p \in C$, and choose a coordinate chart $\phi : U \xrightarrow{\approx} \mathbb{R}^n \times \mathbb{R}^{m-n}$ defined on an open neighborhood U of p such that $\phi(p) = (0,0)$ and $\phi(U \cap C) = \mathbb{R}^n \times \{0\}$. By composing this chart with a diffeomorphism of $\mathbb{R}^n \times \mathbb{R}^{m-n}$ that is constant in the first component, we may assume that the matrix of the Hessian in the direction normal to $\mathbb{R}^n \times \{0\}$ at $(0,0) \in \mathbb{R}^n \times \mathbb{R}^{m-n}$ for the local expression $h(x,y) = (f \circ \phi^{-1})(x,y)$:

$$M_0^\nu(f) = \left(\frac{\partial^2 h}{\partial y_i \partial y_j}\bigg|_{(0,0)} \right)$$

is a diagonal matrix with the first k diagonal entries equal to -1 and the rest equal to $+1$.

The assumption that f is Morse-Bott means that for every $x \in \mathbb{R}^n$ the quadratic form

$$q_x(y) = {}^t y \left(\frac{\partial^2 h}{\partial y_i \partial y_j}\bigg|_{(x,0)} \right) y$$

is non-degenerate. If we fix $x \in \mathbb{R}^n$ and apply Palais' construction in the proof of the Morse Lemma (Lemma 3.11) to the quadratic form $q_x : \mathbb{R}^{m-n} \to \mathbb{R}$, then we get a family ψ_x of diffeomorphisms depending smoothly on x between neighborhoods of $0 \in \mathbb{R}^{m-n}$ such that

$$h(x, \psi_x(y)) = q_x(y).$$

Therefore, $\tilde{\phi}^{-1}(x,y) = \phi^{-1}(x, \psi_x(y)) : \mathbb{R}^n \times \mathbb{R}^{m-n} \to M$ is a chart such that

$$(f \circ \tilde{\phi}^{-1})(x,y) = q_x(y).$$