It's clear that $M_x^\nu(f)$ depends smoothly on x, and hence by Proposition 3.16, for every $x \in \mathbb{R}^n$ sufficiently close to 0 there exists a matrix $Q_x \in GL_{m-n}(\mathbb{R})$ depending smoothly on x such that

$$ {}^tQ_x \left(\left. \frac{\partial^2 h}{\partial y_i \partial y_j} \right|_{(x,0)} \right) Q_x = \left(\left. \frac{\partial^2 h}{\partial y_i \partial y_j} \right|_{(0,0)} \right). $$

Therefore, for all $x \in \mathbb{R}^n$ sufficiently close to 0 we have

$$ \begin{aligned} (f \circ \tilde{\phi}^{-1})(x, Q_x y) &= q_x(Q_x y) \\ &= {}^t y {}^t Q_x \left(\left. \frac{\partial^2 h}{\partial y_i \partial y_j} \right|_{(x,0)} \right) Q_x y \\ &= {}^t y \left(\left. \frac{\partial^2 h}{\partial y_i \partial y_j} \right|_{(0,0)} \right) y \\ &= \sum_{j=1}^{m-n} \delta_j y_j^2 \end{aligned} $$

where $\delta_j = \left. \frac{\partial^2 h}{\partial y_j^2} \right|_{(0,0)} = -1$ for all $j = 1, \ldots, k$ and $\delta_j = \left. \frac{\partial^2 h}{\partial y_j^2} \right|_{(0,0)} = +1$ for all $j = k+1, \ldots, m$.

□

Definition 3.52 *Let $f : M \to \mathbb{R}$ be a Morse-Bott function on a finite dimensional smooth manifold M, and let C be a critical submanifold of f. For any $p \in C$ let λ_p denote the index of $H_p^\nu(f)$. This integer is the dimension of the unstable normal space $\nu_p^- C$, which is locally constant by the preceding lemma. If C is connected, then λ_p is constant throughout C and we call $\lambda_p = \lambda_C$ the* ***Morse-Bott index*** *of the connected critical submanifold C.*

The Morse-Bott inequalities

We will now show how to generalize the Morse inequalities from the previous section. Recall that the Morse polynomial $M_t(f)$ of a Morse function $f : M \to \mathbb{R}$ is

$$ M_t(f) = \sum_{k=0}^{m} \nu_k t^k $$

where ν_k is the number of critical points of index k. The Poincaré polynomial of M is

$$ P_t(M) = \sum_{k=0}^{m} b_k t^k $$