where b_k is the k^{th} Betti number of M. By Theorem 3.36 we have

$$M_t(f) = P_t(M) + (1+t)R(t)$$

where $R(t)$ is a polynomial with non-negative integer coefficients.

Let $f : M \to \mathbb{R}$ be a Morse-Bott function on a finite dimensional compact smooth manifold, and assume that

$$\text{Cr}(f) = \coprod_{j=1}^{l} C_j,$$

where $C_1, \ldots, C_l$ are disjoint connected orientable critical submanifolds. Under these assumptions, we define the Morse-Bott polynomial of f to be

$$MB_t(f) = \sum_{j=1}^{l} P_t(C_j)t^{\lambda_j}$$

where λ_j is the Morse-Bott index of the critical submanifold C_j and $P_t(C_j)$ is the Poincaré polynomial of C_j. Clearly $MB_t(f)$ reduces to $M_t(f)$ when f is a Morse function. For a proof of the following theorem we refer the reader to Section 1 of [8] or [16].

Theorem 3.53 (Morse-Bott Inequalities) *Let $f : M \to \mathbb{R}$ be a Morse-Bott function on a finite dimensional compact smooth manifold, and assume that all the unstable normal bundles $\nu_*^- C$ of the critical submanifolds C are orientable. Then there exists a polynomial $R(t)$ with non-negative integer coefficients such that*

$$MB_t(f) = P_t(M) + (1+t)R(t).$$

Remark 3.54 The preceding theorem can be generalized to the case where the unstable normal bundles are not orientable by using the homology of the critical submanifolds with twisted coefficients in an orientation bundle (see Section 1 of [8]). For an interesting approach to proving the preceding theorem using the Morse Homology Theorem (Theorem 7.4) and spectral sequences see [90]. However, as was pointed out to us by a reader of an early version of this manuscript, [90] fails to take into account the necessary assumption that the unstable normal bundles are orientable.

Examples of the Morse-Bott inequalities

Example 3.55 If $f \equiv c \in \mathbb{R}$ is a constant function, then $MB_t(f)$ is just $P_t(M)$.