Example 3.56 If $g : S^n \to \mathbb{R}$ is given by $g(x_1, \ldots, x_{n+1}) = x_{n+1}^2$ (the square of the height function from Examples 3.5 and 3.49), then

$$\text{Cr}(g) = \{(x_1, \ldots, x_{n+1}) \in S^n |\ x_{n+1} = -1,\ x_{n+1} = 0,\ \text{or } x_{n+1} = 1\}.$$

Each point $x_{n+1} = \pm 1$ has index n and the equator $x_{n+1} = 0$ has index 0. Hence,

$$\begin{aligned} MB_t(g) &= P_t(\{x_{n+1} = 0\})t^0 + P_t(\{x_{n+1} = -1\})t^n + P_t(\{x_{n+1} = +1\})t^n \\ &= P_t(S^{n-1}) + t^n + t^n \\ &= 1 + t^{n-1} + 2t^n. \end{aligned}$$

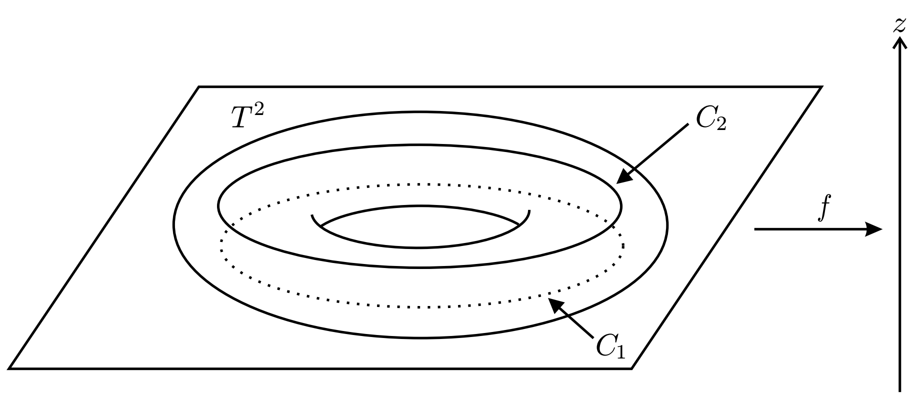
Since $P_t(S^n) = 1 + t^n$ we see that

$$1 + t^{n-1} + 2t^n = (1 + t^n) + (1 + t)t^{n-1},$$

and hence, $R(t) = t^{n-1}$.

Remark 3.57 Note that the preceding example provides a counterexample for Morse-Bott functions to the misstatements discussed in Remark 3.37.

Example 3.58 If $f : T^2 \to \mathbb{R}$ is the height function of a torus lying on plane, then $\text{Cr}(f) = C_1 \cup C_2$ where C_1 is the circle where f takes its minimum and C_2 is the circle where f takes its maximum.



The Morse-Bott index of C_1 is 0 and the Morse-Bott index of C_2 is 1. Since $P_t(C_1) = P_t(C_2) = 1 + t$ we have

$$MB_t(f) = (1 + t)t^0 + (1 + t)t = 1 + 2t + t^2 = P_t(T^2)$$

and $R(t) \equiv 0$.

We would like to thank the reader of an early version of this manuscript who provided us with the following example. This example shows that the