statement of the Morse-Bott inequalities in Theorem 3.53 may not hold when the unstable normal bundles are not orientable. The correct statement of the Morse-Bott inequalities in the case where the unstable normal bundles are not orientable uses homology with local coefficients (see Section 1 of [8]).

Example 3.59 Consider the Morse-Bott function on T^2 described in the previous example. Pull this back to a function on $T^2 \times [0, 1]$. Now glue the ends together by a reflection on T^2 which preserves the Morse-Bott function but reflects each critical circle. We then get a Morse-Bott function on the mapping torus of a reflection on T^2, whose critical submanifolds are Klein bottles, which fails the Morse-Bott inequalities as stated in Theorem 3.53.

A perturbation technique for Morse-Bott functions

We end this section by giving a useful construction relating Morse-Bott functions and Morse functions. We will see in Theorem 5.27 that given any smooth function f on a smooth manifold there exists a Morse function g arbitrarily close to f. However, the proof of Theorem 5.27 is not constructive since it relies on Sard's Theorem. The following useful construction produces an explicit Morse function arbitrarily close to a given Morse-Bott function [14].

Let T_j be a small tubular neighborhood around each connected component $C_j \subseteq \text{Cr}(f)$ for all $j = 1, \ldots, l$. Pick a positive Morse function f_j on each C_j for all $j = 1, \ldots, l$, and extend f_j to a function on T_j by making f_j constant in the direction normal to C_j. Let ρ_j be a bump function which is equal to 1 near C_j and equal to 0 outside of T_j. Choose $\varepsilon > 0$ and define

$$g = f + \varepsilon \left(\sum_{j=1}^{l} \rho_j f_j \right).$$

For appropriate bump functions ρ_j and ε small enough, g is a Morse function close to f, and the critical points of g are exactly the critical points of f_j for all $j = 1, \ldots, l$. It is easy to see that if $p \in C_j$ is a critical point of $f_j : C_j \to \mathbb{R}$ of index λ_p^j, then p is a critical point of g of index

$$\lambda_p^g = \lambda_j + \lambda_p^j$$

where λ_j is the Morse-Bott index of C_j. For more details about this construction see Section 5 of [16].

Problems

1. Show that in local coordinates around a critical point $p \in M$ the matrix for the Hessian $H_p(f)$ is given by the matrix $M_p(f)$ of second partial derivatives.