Morse and Morse-Bott Homology

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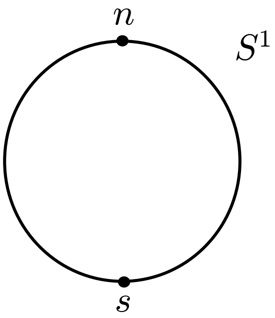
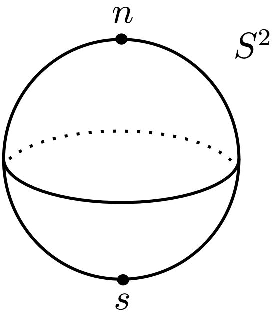
Reference

- David Hurtubise, *Three approaches to Morse-Bott homology*, Afr. Diaspora J. Math. **14** (2012), no. 2, 145–177.

Special volume in honor of Professor Augustin Banyaga on the occasion of his 65th birthday.

Examples for Betti numbers

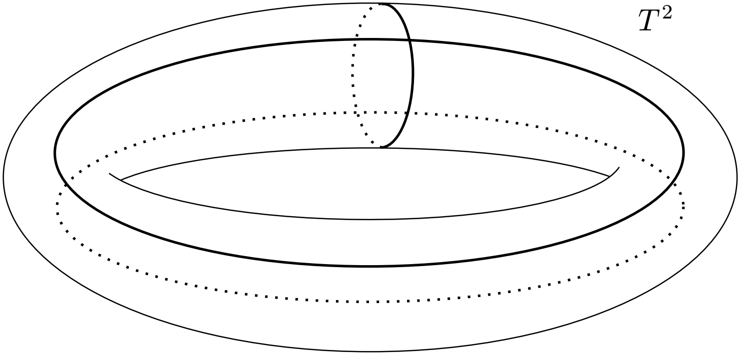




$$\begin{aligned} b_0(S^1) &= 1 \\ b_1(S^1) &= 1 \\ b_2(S^1) &= 0 \end{aligned}$$

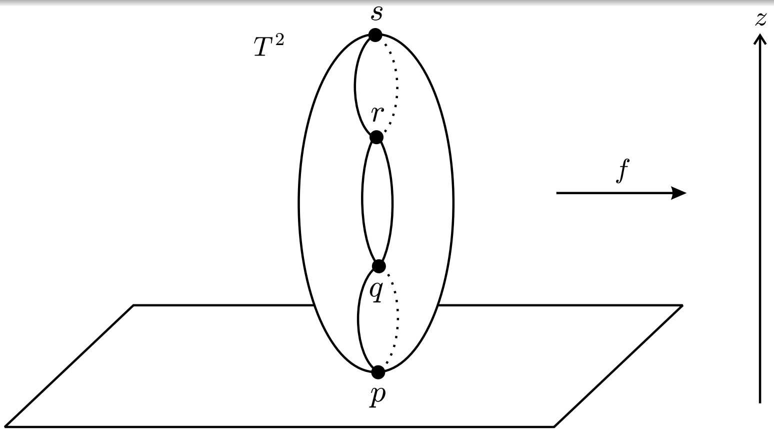
$$\begin{aligned} b_0(S^2) &= 1 \\ b_1(S^2) &= 0 \\ b_2(S^2) &= 1 \end{aligned}$$

Examples for Betti numbers



$$\begin{aligned} b_0(T^2) &= 1 \\ b_1(T^2) &= 2 \\ b_2(T^2) &= 1 \end{aligned}$$

Morse functions



Fundamental Idea: There should be a connection between the critical points of a Morse function $f : M \to \mathbb{R}$ and the Betti numbers of M.

The Morse inequalities

Weak Morse inequalities: $\nu_k(f) \geq b_k(M)$ for all $k = 0, \ldots, m$, where $\nu_k(f) = \#Cr_k(f)$ is the number of critical points of index k.

Strong Morse inequalities:

$$\begin{array}{ccc} \nu_0 & \geq & b_0 \\ \nu_1 - \nu_0 & \geq & b_1 - b_0 \\ \nu_2 - \nu_1 + \nu_0 & \geq & b_2 - b_1 + b_0 \\ \vdots & \vdots & \vdots \\ v_{m-1} - v_{m-2} + \cdots \pm v_0 & \geq & b_{m-1} - b_{m-2} + \cdots \pm b_0 \\ & & \\ v_m - v_{m-1} + v_{m-2} - \cdots \pm v_0 & = & b_m - b_{m-1} + b_{m-2} - \cdots \pm b_0 \end{array}$$

Corollary: $\mathcal{X}(M) = b_0 - b_1 + \cdots \pm b_m = (-1)^m \mathcal{X}(M)$

The polynomial Morse inequalities

The **Poincaré polynomial** of M is defined to be

$$P_t(M) = \sum_{k=0}^{m} b_k(M) t^k$$

and the **Morse polynomial** of $f : M \to \mathbb{R}$ is defined to be

$$M_t(f) = \sum_{k=0}^{m} \nu_k(f) t^k.$$

Theorem (Polynomial Morse Inequalities): For any Morse function $f : M \to \mathbb{R}$ on a smooth manifold M we have

$$M_t(f) = P_t(M) + (1+t)R(t)$$

where $R(t)$ is a polynomial with non-negative integer coefficients.

Stable and unstable manifolds

Let $p \in M$ be a critical point of a smooth function $f : M \to \mathbb{R}$ on a smooth Riemannian manifold M of dimension $m < \infty$, and let $\varphi : \mathbb{R} \times M \to M$ be the 1-parameter family of diffeomorphisms determined by $-\nabla f$. The **stable manifold** of p is

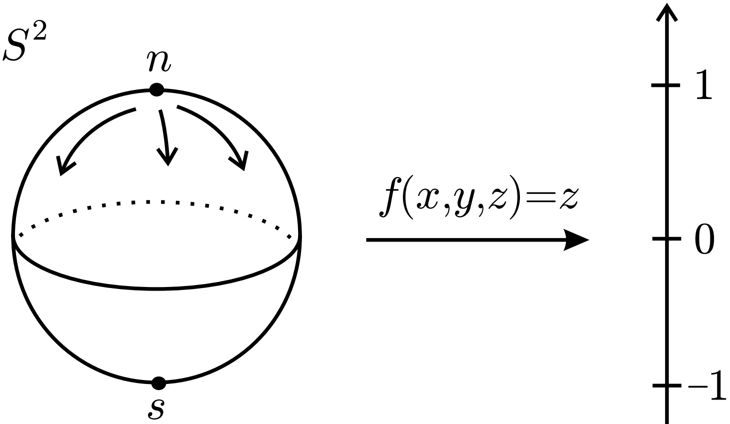
$$W^s(p) = \{x \in M | \lim_{t \to \infty} \varphi_t(x) = p\}$$

and the **unstable manifold** of p is

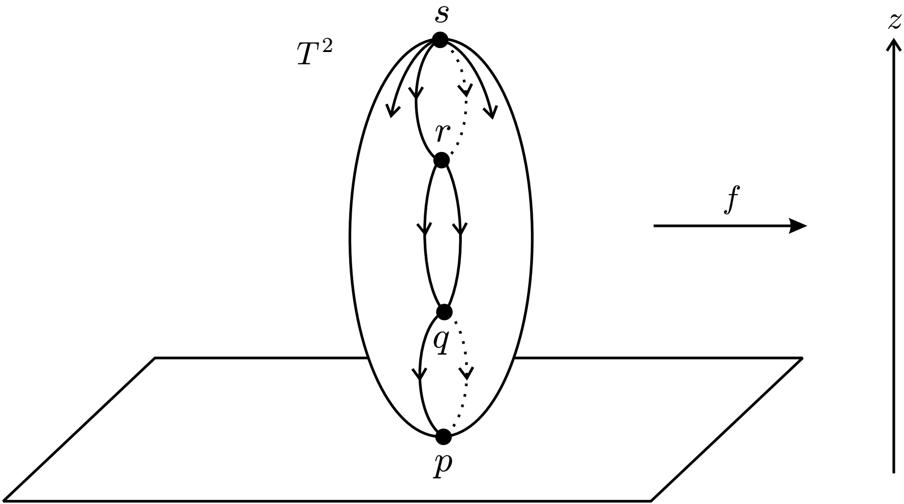
$$W^u(p) = \{x \in M | \lim_{t \to -\infty} \varphi_t(x) = p\}.$$

The Stable/Unstable Manifold Theorem: If p is a nondegenerate critical point, then the stable manifold $W^s(p)$ is a smoothly embedded open disk of dimension $m - \lambda_p$ and the unstable manifold $W^u(p)$ is a smoothly embedded open disk of dimension λ_p.

Examples for stable and unstable manifolds



Examples for stable and unstable manifolds



Morse-Smale transversality

A Morse function $f : M \to \mathbb{R}$ is called **Morse-Smale** if and only if all its stable and unstable manifolds intersect transversally, i.e. $W^u(q) \pitchfork W^s(p)$ for all $p, q \in Cr(f)$.

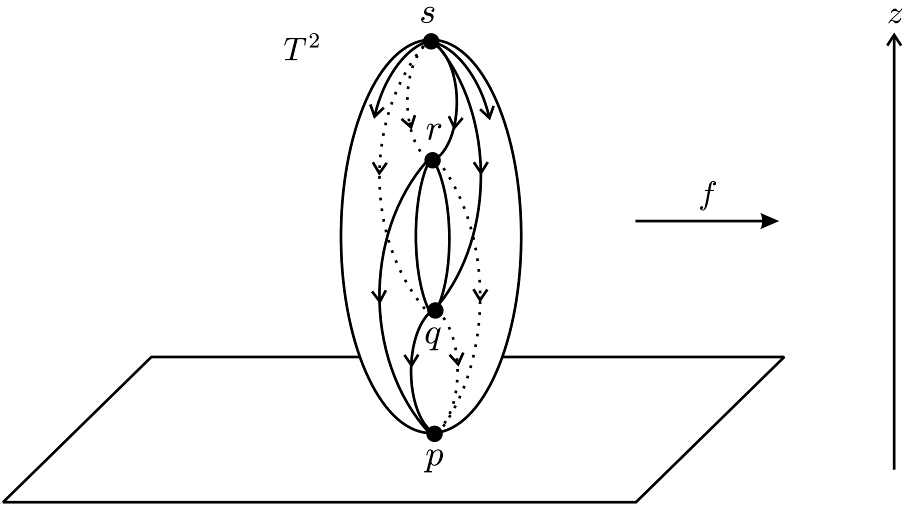
If $W^u(q) \cap W^s(p) \neq \emptyset$, then this condition implies that $W^u(q) \cap W^s(p)$ is a manifold of dimension $\lambda_q - \lambda_p$ and the **moduli space**

$$\mathcal{M}(q,p) = \left(W^u(q) \cap W^s(p)\right)/\mathbb{R}$$

is a manifold of dimension $\lambda_q - \lambda_p - 1$.

Note: The dimension of M does not affect the dimension of the moduli space $\mathcal{M}(q,p)$.

A Morse-Smale function on a 2-torus



The Morse-Smale-Witten chain complex

Let $f : M \to \mathbb{R}$ be a Morse-Smale function on a compact smooth Riemannian manifold M of dimension $m < \infty$, and assume that orientations for the unstable manifolds of f have been chosen. Let $C_k(f)$ be the free abelian group generated by the critical points of index k, and let

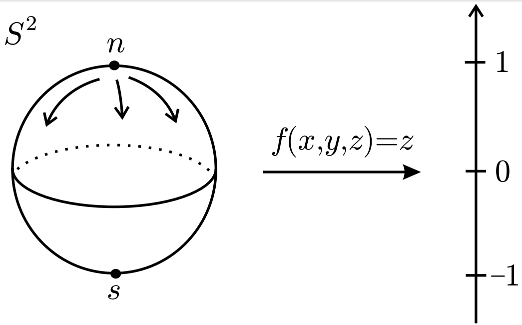
$$C_*(f) = \bigoplus_{k=0}^{m} C_k(f).$$

Define a homomorphism $\partial_k : C_k(f) \to C_{k-1}(f)$ by

$$\partial_k(q) \;=\; \sum_{p \in \mathrm{Cr}_{k-1}(f)} n(q,p)p$$

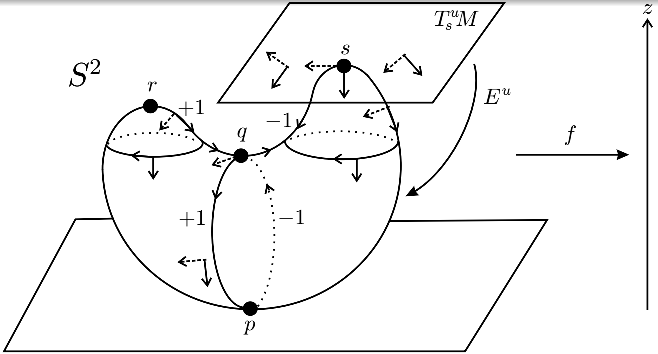
where $n(q,p)$ is the number of gradient flow lines from q to p counted with sign. The pair $(C_*(f), \partial_*)$ is called the **Morse-Smale-Witten chain complex** of f.

The height function on the 2-sphere



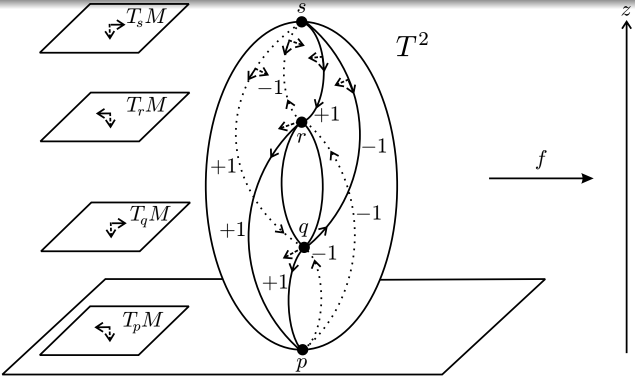
$$\begin{array}{ccccccc}
C_2(f) & \xrightarrow{\partial_2} & C_1(f) & \xrightarrow{\partial_1} & C_0(f) & \longrightarrow & 0 \\
\updownarrow \approx & & \updownarrow \approx & & \updownarrow \approx & & \\
< n > & \xrightarrow{\partial_2} & < 0 > & \xrightarrow{\partial_1} & < s > & \longrightarrow & 0
\end{array}$$

The height function on a deformed 2-sphere



$$\begin{array}{ccccccc} C_2(f) & \xrightarrow{\partial_2} & C_1(f) & \xrightarrow{\partial_1} & C_0(f) & \longrightarrow & 0 \\ \updownarrow \approx & & \updownarrow \approx & & \updownarrow \approx & & \\ < r, s > & \xrightarrow{\partial_2} & < q > & \xrightarrow{\partial_1} & < p > & \longrightarrow & 0 \end{array}$$

The height function on a tilted 2-torus

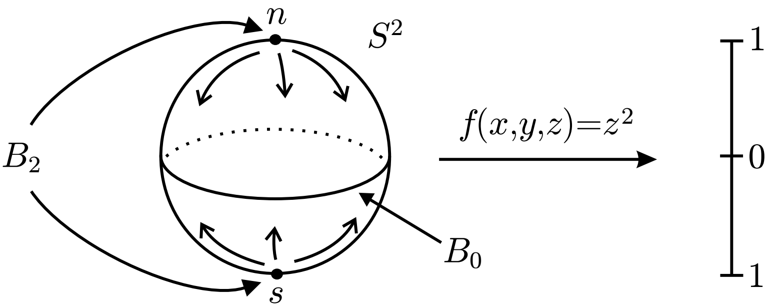


$$
\begin{array}{ccccccc}
C_2(f) & \xrightarrow{\partial_2} & C_1(f) & \xrightarrow{\partial_1} & C_0(f) & \longrightarrow & 0 \\
\updownarrow \approx & & \updownarrow \approx & & \updownarrow \approx & & \\
< s > & \xrightarrow{\partial_2} & < q, r > & \xrightarrow{\partial_1} & < p > & \longrightarrow & 0
\end{array}
$$

References for Morse homology

- Augustin Banyaga and David Hurtubise, **Lectures on Morse homology**, Kluwer Texts in the Mathematical Sciences **29**, Kluwer Academic Publishers Group, 2004.
- John Milnor, **Lectures on the h-cobordism theorem**, Princeton University Press, 1965.
- Liviu Nicolaescu, **An Invitation to Morse Theory**, *Universitext*, Springer 2007.
- Matthias Schwarz, **Morse homology**, Progress in Mathematics **111**, Birkhäuser, 1993.
- Edward Witten, *Supersymmetry and Morse theory*, J. Differential Geom. **17** (1982), no. 4, 661–692.

A Morse-Bott function on the 2-sphere



Can we construct a chain complex for this function? a spectral sequence? a multicomplex?

Generic perturbations

Theorem (Morse 1932)

Let M be a finite dimensional smooth manifold. Given any smooth function $f : M \to \mathbb{R}$ and any $\varepsilon > 0$, there is a Morse function $g : M \to \mathbb{R}$ such that $\sup\{|f(x) - g(x)| \mid x \in M\} < \varepsilon$.

Theorem

Let M be a finite dimensional compact smooth manifold. The space of all C^r Morse functions on M is an open dense subspace of $C^r(M, \mathbb{R})$ for any $2 \leq r \leq \infty$ where $C^r(M, \mathbb{R})$ denotes the space of all C^r functions on M with the C^r topology.

Why not just perturb the Morse-Bott function $f : M \to \mathbb{R}$ to a Morse function?

Fiber bundles and group actions

If $\pi : E \to B$ is a smooth fiber bundle with fiber F, and f is a Morse function on B, then $f \circ \pi$ is a Morse-Bott function with critical submanifolds diffeomorphic to F.

$$\begin{array}{ccccc} F & \longrightarrow & E & & \\ & & \Big\downarrow \pi & & \\ & & B & \xrightarrow{f} & \mathbb{R} \end{array}$$

In particular, if G is a Lie group acting on M, then this might be useful for studying equivariant homology.

$$\begin{array}{ccccc} M & \longrightarrow & EG \times_G M & & \\ & & \Big\downarrow \pi & & \\ & & BG & \xrightarrow{f} & \mathbb{R} \end{array}$$

Symplectic and Instanton Floer homology

- Andreas Floer, *An instanton-invariant for 3-manifolds*, Comm. Math. Phys. **118** (1988), no. 2, 215–240.
- Andreas Floer, *Morse theory for Lagrangian intersections*, Journal of Differential Geom **28** (1988), 513–547.
- Andreas Floer, *Symplectic fixed points and holomorphic spheres*, Comm. Math. Phys. **120** (1989), no. 4, 575–611.

Generalizations: Donaldson polynomials for 4-manifolds with boundary, knot homology groups, comparing the quantum cup product to the pair of pants product.

An explicit perturbation of $f : M \to \mathbb{R}$

Let T_j be a small tubular neighborhood around each connected component $C_j \subseteq Cr(f)$ for all $j = 1, \ldots, l$. Pick a positive Morse function $f_j : C_j \to \mathbb{R}$ and extend f_j to a function on T_j by making f_j constant in the direction normal to C_j for all $j = 1, \ldots, l$.

Let $\tilde{T}_j \subset T_j$ be a smaller tubular neighborhood of C_j with the same coordinates as T_j, and let ρ_j be a smooth bump function which is constant in the coordinates parallel to C_j, equal to 1 on $\tilde{T}_j$, equal to 0 outside of T_j, and decreases on $T_j - \tilde{T}_j$ as the coordinates move away from C_j. For small $\varepsilon > 0$ (and a careful choice of the metric) this determines a Morse-Smale function

$$h_\varepsilon = f + \varepsilon \left(\sum_{j=1}^{l} \rho_j f_j \right).$$

Critical points of the perturbed function

If $p \in C_j$ is a critical point of $f_j : C_j \to \mathbb{R}$ of index λ_p^j, then p is a critical point of h_ε of index

$$\lambda_p^{h_\varepsilon} = \lambda_j + \lambda_p^j$$

where λ_j is the Morse-Bott index of C_j.

Theorem (Morse-Bott Inequalities)

Let $f : M \to \mathbb{R}$ be a Morse-Bott function on a finite dimensional oriented compact smooth manifold, and assume that all the critical submanifolds of f are orientable. Then there exists a polynomial $R(t)$ with non-negative integer coefficients such that

$$MB_t(f) = P_t(M) + (1+t)R(t).$$

(Different orientation assumptions in [Banyaga-H 2009] than the proof using the Thom Isomorphism Theorem.)

The idea behind the Banyaga-H proof

$$
\begin{aligned}
MB_t(f) &= \sum_{j=1}^{l} P_t(C_j)t^{\lambda_j} \\
&= \sum_{j=1}^{l} \left(M_t(f_j) - (1+t)R_j(t) \right) t^{\lambda_j} \\
&= \sum_{j=1}^{l} M_t(f_j)t^{\lambda_j} - (1+t)\sum_{j=1}^{l} R_j(t)t^{\lambda_j} \\
&= M_t(h) - (1+t)\sum_{j=1}^{l} R_j(t)t^{\lambda_j} \\
&= P_t(M) + (1+t)R_h(t) - (1+t)\sum_{j=1}^{l} R_j(t)t^{\lambda_j}
\end{aligned}
$$

Cascades (Frauenfelder 2003 and Bourgeois 2002)

Let $f : M \to \mathbb{R}$ be a Morse-Bott function and suppose

$$\mathrm{Cr}(f) = \coprod_{j=1}^{l} C_j,$$

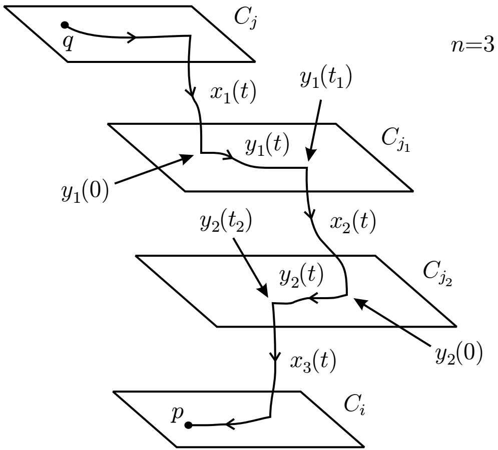
where $C_1, \ldots, C_l$ are disjoint connected critical submanifolds of Morse-Bott index $\lambda_1, \ldots, \lambda_l$ respectively. Let $f_j : C_j \to \mathbb{R}$ be a Morse function on the critical submanifold C_j for all $j = 1, \ldots, l$.

Definition

If $q \in C_j$ is a critical point of the Morse function $f_j : C_j \to \mathbb{R}$ for some $j = 1, \ldots, l$, then the **total index** of q, denoted λ_q, is defined to be the sum of the Morse-Bott index of C_j and the Morse index of q relative to f_j, i.e.

$$\lambda_q = \lambda_j + \lambda_q^j.$$

A 3-cascade



Definition

Denote the space of flow lines from q to p with n cascades by $W_n^c(q,p)$, and denote the quotient of $W_n^c(q,p)$ by the action of $\mathbb{R}^n$ by $\mathcal{M}_n^c(q,p) = W_n^c(q,p)/\mathbb{R}^n$. The **set of unparameterized flow lines with cascades from q to p** is defined to be

$$\mathcal{M}^c(q,p) = \bigcup_{n \in \mathbb{Z}_+} \mathcal{M}_n^c(q,p)$$

where $\mathcal{M}_0^c(q,p) = W_0^c(q,p)/\mathbb{R}$.

The $\mathbb{Z}_2$-cascade chain complex

Define the k^{th} chain group $C_k^c(f)$ to be the free abelian group generated by the critical points of total index k of the Morse-Smale functions f_j for all $j = 1, \ldots, l$, and define $n^c(q, p; \mathbb{Z}_2)$ to be the number of flow lines with cascades between a critical point q of total index k and a critical point p of total index $k - 1$ counted mod 2. Let

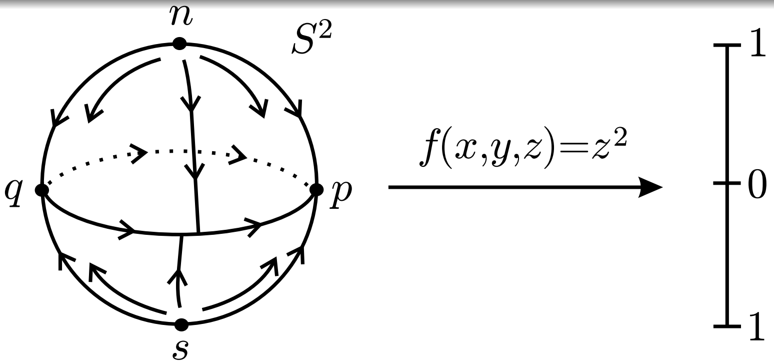
$$C_*^c(f) \otimes \mathbb{Z}_2 = \bigoplus_{k=0}^{m} C_k^c(f) \otimes \mathbb{Z}_2$$

and define a homomorphism $\partial_k^c : C_k^c(f) \otimes \mathbb{Z}_2 \to C_{k-1}^c(f) \otimes \mathbb{Z}_2$ by

$$\partial_k^c(q) = \sum_{p \in Cr(f_{k-1})} n^c(q, p; \mathbb{Z}_2) p.$$

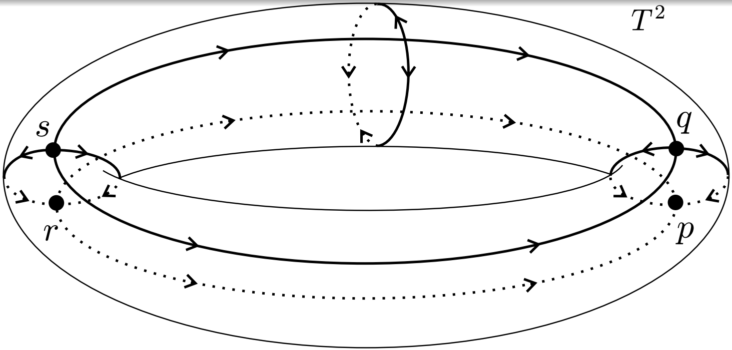
The pair $(C_*^c(f) \otimes \mathbb{Z}_2, \partial_*^c)$ is called the **cascade chain complex** with $\mathbb{Z}_2$ coefficients.

A cascade chain complex for the 2-sphere



$$\begin{array}{ccccccc} C_2^c(f) & \xrightarrow{\partial_2^c} & C_1^c(f) & \xrightarrow{\partial_1^c} & C_0^c(f) & \longrightarrow & 0 \\ \updownarrow \approx & & \updownarrow \approx & & \updownarrow \approx & & \\ < n, s > & \xrightarrow{\partial_2^c} & < q > & \xrightarrow{\partial_1^c} & < p > & \longrightarrow & 0 \end{array}$$

A cascade chain complex for the 2-torus



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< s > & \xrightarrow{\partial_2^c} & < q, r > & \xrightarrow{\partial_1^c} & < p > & \longrightarrow & 0
\end{array}$$

The Arnold-Givental conjecture

Let (M, ω) be a $2n$-dimensional compact symplectic manifold, $L \subset M$ a compact Lagrangian submanifold, and $R \in \text{Diff}(M)$ an antisymplectic involution, i.e. $R^*\omega = -\omega$ and $R^2 = \text{id}$, whose fixed point set is L.

Conjecture. Let H_t be a smooth family of Hamiltonian functions on M for $0 \leq t \leq 1$ and denote by Φ_H the time-1 map of the flow of the Hamiltonian vector field of H_t. If L intersects $\Phi_H(L)$ transversally, then

$$\#(L \cap \Phi_H(L)) \geq \sum_{k=0}^{n} b_k(L; \mathbb{Z}_2).$$

Proved by Frauenfelder for a class of Lagrangians in Marsden-Weinstein quotients by letting $H \to 0$ (2004).

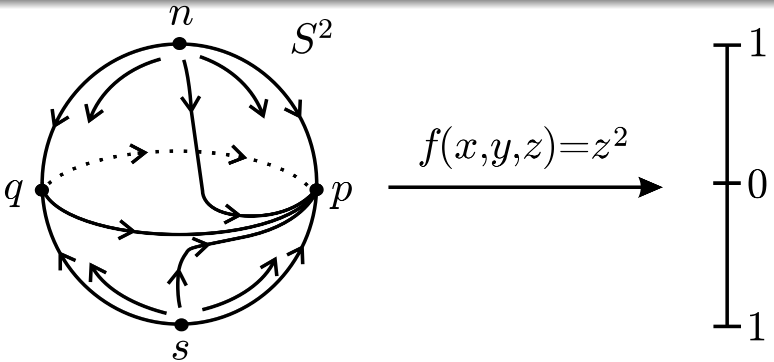
The explicit perturbation of $f : M \to \mathbb{R}$

Let T_j be a small tubular neighborhood around each connected component $C_j \subseteq Cr(f)$ for all $j = 1, \ldots, l$. Pick a positive Morse function $f_j : C_j \to \mathbb{R}$ and extend f_j to a function on T_j by making f_j constant in the direction normal to C_j for all $j = 1, \ldots, l$.

Let $\tilde{T}_j \subset T_j$ be a smaller tubular neighborhood of C_j with the same coordinates as T_j, and let ρ_j be a smooth bump function which is constant in the coordinates parallel to C_j, equal to 1 on $\tilde{T}_j$, equal to 0 outside of T_j, and decreases on $T_j - \tilde{T}_j$ as the coordinates move away from C_j. For small $\varepsilon > 0$ (and a careful choice of the metric) this determines a Morse-Smale function

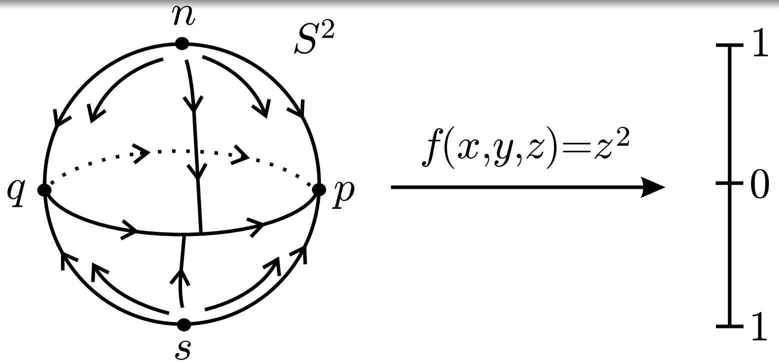
$$h_\varepsilon = f + \varepsilon \left(\sum_{j=1}^{l} \rho_j f_j \right).$$

A perturbed Morse-Bott function on the 2-sphere



$$\begin{array}{ccccccc}
C_2(h_\varepsilon) & \xrightarrow{\partial_2} & C_1(h_\varepsilon) & \xrightarrow{\partial_1} & C_0(h_\varepsilon) & \longrightarrow & 0 \\
\updownarrow \approx & & \updownarrow \approx & & \updownarrow \approx & & \\
< n, s > & \xrightarrow{\partial_2} & < q > & \xrightarrow{\partial_1} & < p > & \longrightarrow & 0
\end{array}$$

A cascade chain complex for the 2-sphere



$$\begin{array}{ccccccc} C_2^c(f) & \xrightarrow{\partial_2^c} & C_1^c(f) & \xrightarrow{\partial_1^c} & C_0^c(f) & \longrightarrow & 0 \\ \updownarrow \approx & & \updownarrow \approx & & \updownarrow \approx & & \\ < n, s > & \xrightarrow{\partial_2^c} & < q > & \xrightarrow{\partial_1^c} & < p > & \longrightarrow & 0 \end{array}$$

Comparing the cascade and Morse chain complexes

For every sufficiently small $\varepsilon > 0$ and $k = 0, \ldots, m$ we have

$$C_k^c(f) \approx C_k(h_\varepsilon) = \bigoplus_{\lambda_j + n = k} C_n(f_j).$$

Is $\mathcal{M}^c(q,p) \approx \mathcal{M}_{h_\varepsilon}(q,p)$ when $\lambda_q - \lambda_p = 1$?

If so, then we can use the orientations on $\mathcal{M}_{h_\varepsilon}(q,p)$ to define the cascade chain complex over $\mathbb{Z}$ so that $\partial_k^c = -\partial_k$ for all $k = 0, \ldots, m$, where ∂_k is the Morse-Smale-Witten boundary operator of h_ε. In particular,

$$H_*((C_*^c(f), \partial_*^c)) \approx H_*(M; \mathbb{Z}).$$

Theorem (Banyaga-H 2013)

Assume that f satisfies the Morse-Bott-Smale transversality condition with respect to the Riemannian metric g on M, $f_k : C_k \to \mathbb{R}$ satisfies the Morse-Smale transversality condition with respect to the restriction of g to C_k for all $k = 1, \ldots, l$, and the unstable and stable manifolds $W^u_{f_j}(q)$ and $W^s_{f_i}(p)$ are transverse to the beginning and endpoint maps.

1. *When $n = 0, 1$ the set $\mathcal{M}^c_n(q,p)$ is either empty or a smooth manifold without boundary.*
2. *For $n > 1$ the set $\mathcal{M}^c_n(q,p)$ is either empty or a smooth manifold with corners.*
3. *The set $\mathcal{M}^c(q,p)$ is either empty or a smooth manifold without boundary.*

In each case the dimension of the manifold is $\lambda_q - \lambda_p - 1$. The above manifolds are orientable when M and C_k are orientable.

Correspondence of moduli spaces

Theorem (Banyaga-H 2013)

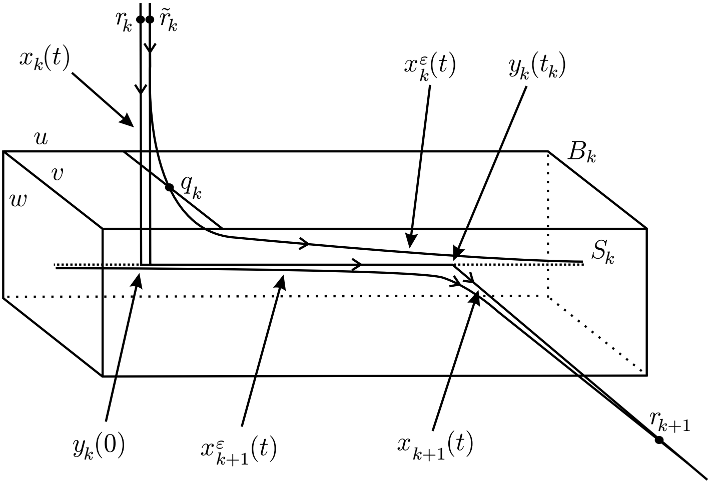
Let $p, q \in$ Cr(h_ε) with $\lambda_q - \lambda_p = 1$. For any sufficiently small $\varepsilon > 0$ there is a bijection between unparameterized cascades and unparameterized gradient flow lines of the Morse-Smale function $h_\varepsilon : M \to \mathbb{R}$ between q and p,

$$\mathcal{M}^c(q, p) \leftrightarrow \mathcal{M}_{h_\varepsilon}(q, p).$$

Definition

Let $p, q \in$ Cr(h_ε) with $\lambda_q - \lambda_p = 1$, define an orientation on the zero dimensional manifold $\mathcal{M}^c(q, p)$ by identifying it with the left hand boundary of $\mathcal{M}_{h_\varepsilon}(q, p) \times [0, \varepsilon]$.

Main idea: The Exchange Lemma



Correspondence of chain complexes

Theorem (Banyaga-H 2011)

For $\varepsilon > 0$ sufficiently small we have $C_k^c(f) = C_k(h_\varepsilon)$ and $\partial_k^c = -\partial_k$ for all $k = 0, \ldots, m$, where ∂_k denotes the Morse-Smale-Witten boundary operator determined by the Morse-Smale function h_ε. In particular, $(C_*^c(f), \partial_*^c)$ is a chain complex whose homology is isomorphic to the singular homology $H_*(M;\mathbb{Z})$.

Moral: The cascade chain complex of a Morse-Bott function $f : M \to \mathbb{R}$ is the same as the Morse-Smale-Witten complex of a small perturbation of f.

Multicomplexes

Let R be a principal ideal domain. A first quadrant **multicomplex** X is a bigraded R-module $\{X_{p,q}\}_{p,q\in\mathbb{Z}_+}$ with differentials

$$\mathrm{d}_i : X_{p,q} \to X_{p-i,q+i-1} \quad \text{for all } i = 0, 1, \ldots$$

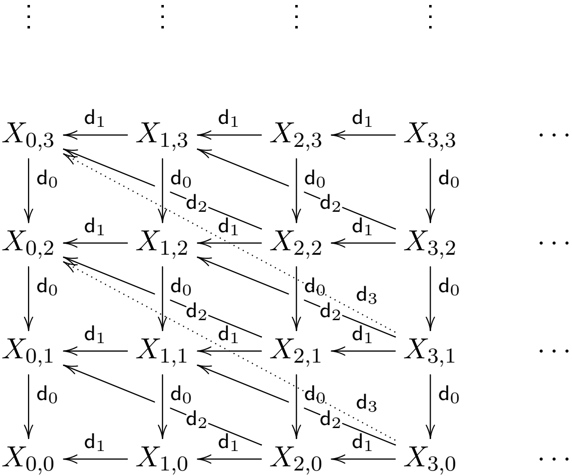
that satisfy

$$\sum_{i+j=n} \mathrm{d}_i \mathrm{d}_j = 0 \quad \text{for all } n.$$

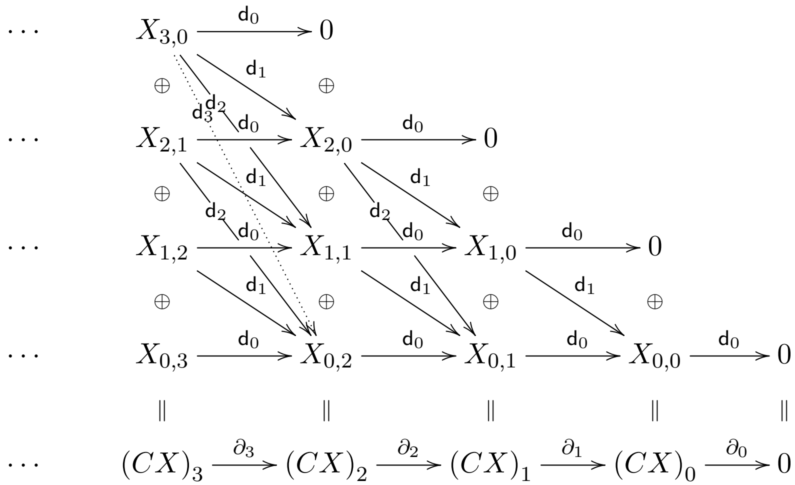
A first quadrant multicomplex can be **assembled** to form a filtered chain complex $((CX)_*, \partial)$ by summing along the diagonals, i.e.

$$(CX)_n \equiv \bigoplus_{p+q=n} X_{p,q} \quad \text{and} \quad F_s(CX)_n \equiv \bigoplus_{\substack{p+q=n \\ p\leq s}} X_{p,q}$$

and $\partial_n = \mathrm{d}_0 \oplus \cdots \oplus \mathrm{d}_n$ for all $n \in \mathbb{Z}_+$. The above relations then imply that $\partial_n \circ \partial_{n+1} = 0$ and $\partial_n(F_s(CX)_*) \subseteq F_s(CX)_*$.



A bicomplex has two filtrations, but a general multicomplex only has one filtration.


$X_{3,0}$
$X_{2,1}$
$X_{2,0}$
$X_{1,2}$
$X_{1,1}$
$X_{1,0}$
$X_{0,3}$
$X_{0,2}$
$X_{0,1}$
$X_{0,0}$
d_0
d_1
d_2
d_3
$(CX)_3$
$(CX)_2$
$(CX)_1$
$(CX)_0$
∂_3
∂_2
∂_1
∂_0

The bigraded module associated to the filtration

$$F_s(CX)_n \equiv \bigoplus_{\substack{p+q=n \\ p \leq s}} X_{p,q}$$

is

$$G((CX)_*)_{s,t} = F_s(CX)_{s+t}/F_{s-1}(CX)_{s+t} \approx X_{s,t}$$

for all $s, t \in \mathbb{Z}_+$, and the E^1 term of the associated spectral sequence is given by

$$E^1_{s,t} = Z^1_{s,t} \big/ \left(Z^0_{s-1,t+1} + \partial Z^0_{s,t+1}\right)$$

where

$$\begin{aligned} Z^1_{s,t} &= \{c \in F_s(CX)_{s+t} | \, \partial c \in F_{s-1}(CX)_{s+t-1}\} \\ Z^0_{s,t} &= \{c \in F_s(CX)_{s+t} | \, \partial c \in F_s(CX)_{s+t-1}\} = F_s(CX)_{s+t}. \end{aligned}$$

$E^1_{s,t}$ and d^1 are induced from d_0 and d_1

Theorem

Let $(\{X_{p,q}\}_{p,q\in\mathbb{Z}_+}, \{\mathsf{d}_i\}_{i\in\mathbb{Z}_+})$ *be a first quadrant multicomplex and* $((CX)_*, \partial)$ *the associated assembled chain complex. Then the* E^1 *term of the spectral sequence associated to the filtration of* $(CX)_*$ *determined by the restriction* $p \leq s$ *is given by* $E^1_{s,t} \approx H_{s+t}(X_{s,*}, \mathsf{d}_0)$ *where* $(X_{s,*}, \mathsf{d}_0)$ *denotes the following chain complex.*

$$\cdots \xrightarrow{\mathsf{d}_0} X_{s,3} \xrightarrow{\mathsf{d}_0} X_{s,2} \xrightarrow{\mathsf{d}_0} X_{s,1} \xrightarrow{\mathsf{d}_0} X_{s,0} \xrightarrow{\mathsf{d}_0} 0$$

Moreover, the d^1 *differential on the* E^1 *term of the spectral sequence is induced from the homomorphism* d_1 *in the multicomplex.*

However, d^r is not induced from d_r for $r \geq 2$

Consider the following first quadrant double complex

$$
\begin{array}{ccccc}
0 & \xleftarrow{\mathsf{d}_1} & 0 & \xleftarrow{\mathsf{d}_1} & 0 \\
\downarrow \mathsf{d}_0 & & \downarrow \mathsf{d}_0 & & \downarrow \mathsf{d}_0 \\
< x_{0,1} > & \xleftarrow{\mathsf{d}_1} & < x_{1,1} > & \xleftarrow{\mathsf{d}_1} & 0 \\
\downarrow \mathsf{d}_0 & & \downarrow \mathsf{d}_0 & & \downarrow \mathsf{d}_0 \\
0 & \xleftarrow{\mathsf{d}_1} & < x_{1,0} > & \xleftarrow{\mathsf{d}_1} & < x_{2,0} >
\end{array}
$$

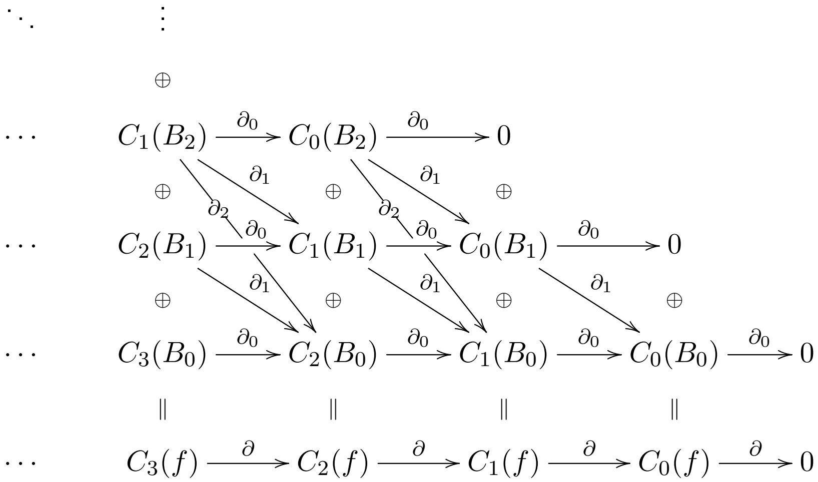
where $< x_{p,q} >$ denotes the free abelian group generated by $x_{p,q}$, the groups $X_{p,q} = 0$ for $p + q > 2$, and the homomorphisms d_0 and d_1 satisfy the following: $\mathsf{d}_0(x_{1,1}) = x_{1,0}$, $\mathsf{d}_1(x_{1,1}) = x_{0,1}$, and $\mathsf{d}_1(x_{2,0}) = x_{1,0}$. In this case, $\mathsf{d}_2 = 0$ but $d^2 \neq 0$

The Morse-Bott-Smale multicomplex

Let $C_p(B_i)$ be the group of "p-dimensional chains" in the critical submanifolds of index i. Assume that $f : M \to \mathbb{R}$ is a Morse-Bott-Smale function and the manifold M, the critical submanifolds, and their negative normal bundles are all orientable.

If $\sigma : P \to B_i$ is a singular C_p-space in $S_p^\infty(B_i)$, then for any $j = 1, \ldots, i$ composing the projection map π_2 onto the second component of $P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j})$ with the endpoint map $\partial_+ : \overline{\mathcal{M}}(B_i, B_{i-j}) \to B_{i-j}$ gives a map

$$P \times_{B_i} \overline{\mathcal{M}}(B_i, B_{i-j}) \xrightarrow{\pi_2} \overline{\mathcal{M}}(B_i, B_{i-j}) \xrightarrow{\partial_+} B_{i-j}.$$


$C_1(B_2) \xrightarrow{\partial_0} C_0(B_2) \xrightarrow{\partial_0} 0$
∂_1 ∂_2
$C_2(B_1) \xrightarrow{\partial_0} C_1(B_1) \xrightarrow{\partial_0} C_0(B_1) \xrightarrow{\partial_0} 0$
$C_3(B_0) \xrightarrow{\partial_0} C_2(B_0) \xrightarrow{\partial_0} C_1(B_0) \xrightarrow{\partial_0} C_0(B_0) \xrightarrow{\partial_0} 0$
$C_3(f) \xrightarrow{\partial} C_2(f) \xrightarrow{\partial} C_1(f) \xrightarrow{\partial} C_0(f) \xrightarrow{\partial} 0$

The Morse-Bott Homology Theorem

Theorem (Banyaga-H 2010)

The homology of the Morse-Bott-Smale multicomplex $(C_*(f), \partial)$ *is independent of the Morse-Bott-Smale function* $f : M \to \mathbb{R}$. *Therefore,*

$$H_*(C_*(f), \partial) \approx H_*(M; \mathbb{Z}).$$

Note: If f is constant, then $(C_*(f), \partial)$ is the chain complex of singular N-cube chains. If f is Morse-Smale, then $(C_*(f), \partial)$ is the Morse-Smale-Witten chain complex. This gives a new proof of the Morse Homology Theorem.

References

- David Austin and Peter Braam, *Morse-Bott theory and equivariant cohomology*, **The Floer memorial volume**, Progr. Math. **133** (1995), 123–183.
- David Austin and Peter Braam, *Equivariant Floer theory and gluing Donaldson polynomials*, Topology **35** (1996), no. 1, 167–200.
- Augustin Banyaga and David Hurtubise, *A proof of the Morse-Bott Lemma*, Expo. Math. **22**, no. 4, 365–373, 2004.
- Augustin Banyaga and David Hurtubise, **Lectures on Morse homology**, Kluwer Texts in the Mathematical Sciences **29**, Springer 2004.
- Augustin Banyaga and David Hurtubise, *The Morse-Bott inequalities via a dynamical systems approach*, Ergodic Theory Dynam. Systems **29** (2009), no. 6, 1693–1703.

References

- Augustin Banyaga and David Hurtubise, *Morse-Bott homology*, Trans. Amer. Math. Soc. **362** (2010), no. 8, 3997–4043.
- Augustin Banyaga and David Hurtubise, *Cascades and perturbed Morse-Bott functions*, Algebr. Geom. Topol. **13** (2013), 237-275.
- Frédéric Bourgeois and Alexandru Oancea, *Symplectic homology, autonomous Hamiltonians, and Morse-Bott moduli spaces*, Duke Math. J. **146** (2009), no. 1, 74–174.
- Frédéric Bourgeois and Alexandru Oancea, *An exact sequence for contact and symplectic homology*, Invent. Math. **179** (2009), no. 3, 611–680.
- Andreas Floer, *An instanton-invariant for* 3*-manifolds*, Comm. Math. Phys. **118** (1988), no. 2, 215–240.

References

- Andreas Floer, *An instanton-invariant for 3-manifolds*, Comm. Math. Phys. **118** (1988), no. 2, 215–240.
- Andreas Floer, *Instanton homology, surgery, and knots*, **Geometry of low-dimensional manifolds**, London Math. Soc. Lecture Note Ser. **150** (1990), 97–114.
- Andreas Floer, *Morse theory for Lagrangian intersections*, Journal of Differential Geom **28** (1988), 513–547.
- Andreas Floer, *Witten's complex and infinite-dimensional Morse theory*, J. Differential Geom. **30** (1989), no. 1, 207–221.
- Urs Frauenfelder, *The Arnold-Givental conjecture and moment Floer homology*, Int. Math. Res. Not. **2004**, no. 42, 2179–2269.

References

- Kenji Fukaya, *Floer homology of connected sum of homology* 3*-spheres*, Topology **35** (1996), no. 1, 89–136.
- David Hurtubise, *Multicomplexes and spectral sequences*, J. Algebra Appl. **9** (2010), no. 4, 519–530.
- David Hurtubise, *Three approaches to Morse-Bott homology*, Afr. Diaspora J. Math. **14** (2012), no. 2, 145–177.
- Jean-Pierre Meyer, *Acyclic models for multicomplexes*, Duke Math. J. **45** (1978), no. 1, 67–85.
- Liviu Nicolaescu, **An Invitation to Morse Theory**, *Universitext*, Springer 2007.

References

- Yongbin Ruan and Gang Tian, *Bott-type symplectic Floer cohomology and its multiplication structures*, Math. Res. Lett. **2** (1995), no. 2, 203–219.
- Jan Swoboda, *Morse homology for the Yang-Mills gradient flow*, arXiv:1103.0845v1, 2011.
- Gang Liu and Gang Tian, *On the equivalence of multiplicative structures in Floer homology and quantum homology*, Acta Math. Sin. **15** (1999), no. 1, 53–80.
- Joa Weber, *The Morse-Witten complex via dynamical systems*, Expo. Math. **24** (2006), no. 2, 127–159.
- Edward Witten. **Supersymmetry and Morse theory**, J. Differential Geom. **17** (1982), no. 4, 661–692 (1983).

The Chern-Simons functional

Let $P \to N$ be a (trivial) principal $SU(2)$-bundle over an oriented closed 3-manifold N, and let $\mathcal{A}$ be the space of connections on P. Define $CS : \mathcal{A} \to \mathbb{R}$ by

$$CS(A) = \frac{1}{4\pi^2} \int_M tr(\frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A).$$

The above functional descends to a function $cs : \mathcal{A}/\mathcal{G} \to \mathbb{R}/\mathbb{Z}$ whose critical points are gauge equivalence classes of flat connections. Extending everything to $P \times \mathbb{R} \to N \times \mathbb{R}$, the gradient flow equation becomes the instanton equation

$$F + *F = 0,$$

where F denotes the curvature and $*$ is the Hodge star operator.

Instanton homology

Andreas Floer, *An instanton-invariant for 3-manifolds*, Comm. Math. Phys. **118** (1988), no. 2, 215–240.

Theorem. When N is a homology 3-sphere the Chern-Simons functional can be perturbed so that it has discrete critical points and defines $\mathbb{Z}_8$-graded homology groups $I_*(N)$ analogous to the Morse homology groups.

Generalizations: Donaldson polynomials for 4-manifolds with boundary, knot homology groups

The symplectic action functional

Let (M, ω) be a closed symplectic manifold and $S^1 = \mathbb{R}/\mathbb{Z}$. A time-dependent Hamiltonian $H : M \times S^1 \to \mathbb{R}$ determines a time-dependent vector field X_H by

$$\omega(X_H(x,t), v) = v(H)(x,t) \text{ for } v \in T_xM.$$

Let $\mathcal{L}(M)$ be the space of free contractible loops on M and

$$\tilde{\mathcal{L}}(M) = \{(x,u) | x \in \mathcal{L}(M), u : D^2 \to M \text{ such that } u(e^{2\pi i t}) = x(t)\}/\sim$$

its universal cover with covering group $\pi_2(M)$. The symplectic action functional $a_H : \tilde{\mathcal{L}}(M) \to \mathbb{R}$ is defined by

$$a_H((x,u)) = \int_{D^2} u^*\omega + \int_0^1 H(x(t),t)\, dt.$$

The Arnold conjecture

Andreas Floer, *Symplectic fixed points and holomorphic spheres*, Comm. Math. Phys. **120** (1989), no. 4, 575–611.

Theorem. Let (P, ω) be a compact symplectic manifold. If I_ω and I_c are proportional, then the fixed point set of every exact diffeomorphism of (P, ω) satisfies the Morse inequalities with respect to any coefficient ring whenever it is nondegenerate.

Generalizations: Allowing H to be degenerate (e.g. $H = 0$) leads to critical submanifolds and Morse-Bott homology.

The Yang-Mills gradient flow

Let (Σ, g) be a closed oriented Riemann surface, G a compact Lie group, $\mathfrak{g}$ its Lie algebra, and P a principal G-bundle over Σ. Pick an ad-invariant inner product on $\mathfrak{g}$, let $\mathcal{A}(P)$ denote the affine space of $\mathfrak{g}$-valued connection 1-forms on P, and define $\mathcal{YM} : \mathcal{A}(P) \to \mathbb{R}$ by

$$\mathcal{YM}(A) = \int_\Sigma F_A \wedge *F_A$$

where $F_A = dA + \frac{1}{2}[A \wedge A]$ is the curvature of A.

The Yang-Mills function is a Morse-Bott function studied by Atiyah-Bott and by Swoboda (2011) using cascades.

Closed Reeb orbits

Let M be a compact, orientable manifold of dimension $2n-1$ with contact form α. The **Reeb vector field** R_α associated to the contact form α is characterized by

$$\begin{aligned} d\alpha(R_\alpha, -) &= 0 \\ \alpha(R_\alpha) &= 1. \end{aligned}$$

Closed trajectories of the Reeb vector field are critical points of the action functional $\mathcal{A} : C^\infty(S^1, M) \to \mathbb{R}$

$$\mathcal{A}(\gamma) = \int_\gamma \alpha.$$

Lemma. For any contact structure ξ on M, there exists a contact form α for ξ such that all closed orbits of R_α are nondegenerate.

Contact homology

Let $\mathbf{A}$ be the graded supercommutative algebra freely generated by the "good" closed Reeb orbits over the graded ring $\mathbb{Q}[H_2(M;\mathbb{Z})/\mathcal{R}]$, i.e. $\gamma_1\gamma_2 = (-1)^{|\gamma_1||\gamma_2|}\gamma_2\gamma_1$.

Theorem. (Eliashberg-Hofer 2000) There is a differential $d : \mathbf{A} \to \mathbf{A}$ defined by counting J-holomorphic curves in the symplectization $(\mathbb{R} \times M, d(e^t\alpha))$ such that $(\mathbf{A}, d)$ is a differential graded algebra. Moreover, $HC_*(M,\xi) \overset{\text{def}}{=} H_*(\mathbf{A}, d)$ is an invariant of the contact structure ξ.

Theorem. (Bourgeois 2002) Assume that α is a contact form of Morse-Bott type for (M,ξ) and that J is an almost complex structure on the symplectization that is S^1-invariant along the critical submanifolds N_T. Then there is a chain complex with a boundary operator defined by counting cascades whose homology is isomorphic to the contact homology $HC_*(M,\xi)$.

Viterbo's symplectic homology

Definition

A compact symplectic manifold (W, ω) has **contact type** boundary if and only if there exists a vector field X defined in a neighborhood of $M = \partial W$ transverse and pointing outward along M such that $\mathcal{L}_X \omega = \omega$.

In this case, $\lambda = \omega(X, \cdot)|_M$ is a contact form on M, and the symplectic homology of W combines the 1-periodic orbits of a Hamiltonian on W with the Reeb orbits on $M = \partial W$.

Bourgeois and Oancea have defined the cascade chain complex for a time-independent Hamiltonian on W whose 1-periodic orbits are transversally nondegenerate (2009). They have also proved that there is an exact sequence relating the symplectic homology groups of W with the linearized contact homology groups of M (2009).

Compactness

Denote the space of nonempty closed subsets of $M \times \overline{\mathbb{R}}^l$ in the topology determined by the Hausdorff metric by $\mathcal{P}^c(M \times \overline{\mathbb{R}}^l)$, and map a broken flow line with cascades $(v_1, \ldots, v_n)$ to its image $\text{Im}(v_1, \ldots, v_n) \subset M$ and the time t_j spent flowing along or resting on each critical submanifold C_j for all $j = 1, \ldots, l$.

Theorem (Banyaga-H 2013)

The space $\overline{\mathcal{M}}^c(q,p)$ of broken flow lines with cascades from q to p is compact, and there is a continuous embedding

$$\mathcal{M}^c(q,p) \hookrightarrow \overline{\mathcal{M}}^c(q,p) \subset \mathcal{P}^c(M \times \overline{\mathbb{R}}^l).$$

Hence, every sequence of unparameterized flow lines with cascades from q to p has a subsequence that converges to a broken flow line with cascades from q to p.